



An Optimal Distribution of Stiffness Over the Height of Shear Buildings to Minimize the Seismic Input Energy

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ABSTRACT

Based on Housner's assumption, the average input energy from earthquakes to a building modeled as a single degree of freedom (SDOF) system, is related mainly to total mass of the building. Thus, based on the above premise for low damping and relatively long period systems, the seismic input energy per unit mass of the system (SDOF or MDOF) is mainly related to the ground motion features. The present study attempts to analytically reveal the range of validity of these assumptions in linear systems and to find an optimal stiffness distribution over the height of high-rise shear linear buildings to minimize the seismic input energy. To accomplish this objective, it is shown from the spectral standpoint that input energy spectra generally is a function of the natural period of vibration, so the input energy is further related to the stiffness of structure, the mass, damping ratio and ground motion characteristics. Subsequently, it is demonstrated that for low to moderate height (up to 20 stories) shear type structures, the optimal distribution of stiffness obeys a parabolic form, while for taller structures, this form is a bell-shaped function.

Keywords:

Seismic input energy;
Linear systems;
Modal analyses;
Input energy spectrum;
Velocity spectrum

1. Introduction

Housner was the first one who used the concept of input energy as seismic design criteria [1]. He presented his pioneering study in the 1st WCEE in 1956. The three main conclusions of his research are of special concern in this study:

- 1) The seismic energy input to a SDOF structure with specified damping, looking from "spectral" or "average" standpoint, is basically constant and independent on its period, especially for low damping ratios.
- 2) Seismic design of structures can be understood as satisfying the following inequality: Energy absorption capacity > seismic input energy. On the other hand, the amount of energy input to an elastic system is the upper bound of energy input to hysteretic systems with the same linear properties. Therefore, seismic design of structures does not mean that it provides too strong elements

with the capability of converting kinetic energy of structure to elastic strain energy; and as an alternative, it is adequate to supply sufficient "capacity" of energy absorption via plastic deformations in structural elements.

- 3) Seismic energy input to a MDOF system basically depends on its "total mass"; therefore, it is equal to energy input to an equivalent SDOF system with the same mass and main period of vibration.

Based on Housner's study, Akiyama published his highly important book of "Earthquake Resistant Limit State Design for Buildings" in 1985. He expanded Housner's assumptions and pointed out their limitations and strong points [2]. He developed input energy spectrum for different site soils. Those spectrums are basically constant with respect to the period of vibration except for the periods smaller

than the predominant periods for the ground. As Housner, Akiyama attempted to simplify seismic design of structures by presuming and demonstrating that “input energy to structures is related mainly to earthquake excitation but scarcely to structural features”. Most of researchers adapted this assumption and equation proposed by Kuwamura and Galambos [3], Fajfar et al [4], Uang and Bertero [5], and Kuwamura et al [6] to establish the earthquake input energy which are merely based on the ground motion characteristics.

Parallel to the research projects in estimating the energy “demand”, other researchers focused on the mechanism of dissipation of the input energy in structural elements by the hysteretic action. Ang and Park [7] related their damage index to the energy dissipation via hysteretic loops. This means that reduction in the hysteretic dissipation of the input energy which is a fraction of the total input energy reduces the structural damage. Thus, it raises an important question:

Is it possible to minimize the seismic input energy to structures by a specific design pattern?

To answer this question, the classical approach to the “input energy demand” problem must be reexamined; the task which is the main purpose of this study.

Various aspects of seismic input energy and its calculation, such as absolute and relative energies, and time interval for integration of related equations have been discussed in the literature [3, 5, 8]. In this research, several basic assumptions and definitions which are widely utilized in the literature are adopted, as follows:

- 1) Relative, rather than absolute input energy is studied.
- 2) Input energy is defined as the energy imposed to the structure by strong ground motion from the beginning ($t=0$) of motion to the end of it ($t=t_d$). Note that definitions of “beginning” and “ending” moments of ground motion are not unique in the literature; however, this issue is not significant in this research. In this study, the “beginning” and “end” of motion are assumed to coincide with the beginning and the end of the record. Furthermore, it is demonstrated that the maximum input energy can be attained not necessarily at the end of motion [5]. However, as mentioned earlier, the input energy of the system at the ending moment

of ground motion is considered as seismic input energy. Thus, the seismic input energy to a *SDOF* system with mass m , frequency ω and damping ratio ζ is defined mathematically as [2]:

$$E_1(m, \omega, \zeta) = -\int_0^{t_d} m\ddot{y}_g \dot{y} dt \tag{1}$$

Where \ddot{y}_g and \dot{y} are ground acceleration and system relative velocity, respectively. For a system with unit mass, Eq. (1) can be written as:

$$E_1(1, \omega, \zeta) = -\int_0^{t_d} \ddot{y}_g \dot{y} dt \tag{2}$$

It is helpful to use an equivalent velocity V_E [2], defined based on the input energy, as:

$$V_E = \sqrt{\frac{2E_1}{m}} \tag{3}$$

where E_1 is input energy to the *SDOF*.

By applying the aforementioned assumptions, in the proceeding sections of this paper, several input energy spectra are primarily obtained; subsequently, the possibility of existence of an optimal stiffness distribution is mathematically demonstrated.

2. Input Energy Spectra

Based on the definition of the input energy, provided in the previous section, several input energy spectra have been obtained by utilizing ten typical earthquake ground motion records as illustrated in Table (1). All of these records have been extracted from *PEER STRONG MOTION DATABASE*.

All records have been normalized to 1.0g. Figures (1) to (3) indicates equivalent velocity V_E spectra versus period of vibration for $\zeta=0, 0.05$, and 0.10, respectively. Design input energy spectrum

Table 1. List of records used to obtain input energy spectra.

Name	Location	Date	Duration (Sec.)	PGA (g)
Chi-Chi, W	Taiwan	09/20/99	89.99	0.9675
Duzci, 90	Turkey	11/12/99	55.89	0.8224
El Centro, 180	USA	5/19/40	39.99	0.3129
Gazli, 90	Uzbekistan	5/17/76	16.20	0.7175
Kobe, 00	Japan	01/16/95	47.98	0.8213
Landers, 00	USA	6/28/92	41.91	0.7848
Loma Prieta, 00	USA	10/18/89	39.94	0.6437
Manjil, Long.	Iran	06/20/90	53.50	0.5146
Northridge, 142	USA	1/17/94	39.99	0.6125
Tabas, Tr.	Iran	09/16/78	32.82	0.8518

(DIES) proposed by Akiyama [2], is further illustrated in the figures. It should be mentioned that the presented DIES values are for highly stiff site soil and

damping ratio of %10. Individual input energy spectra for each record are displayed in Figure (4) for damping ratio of 10%.

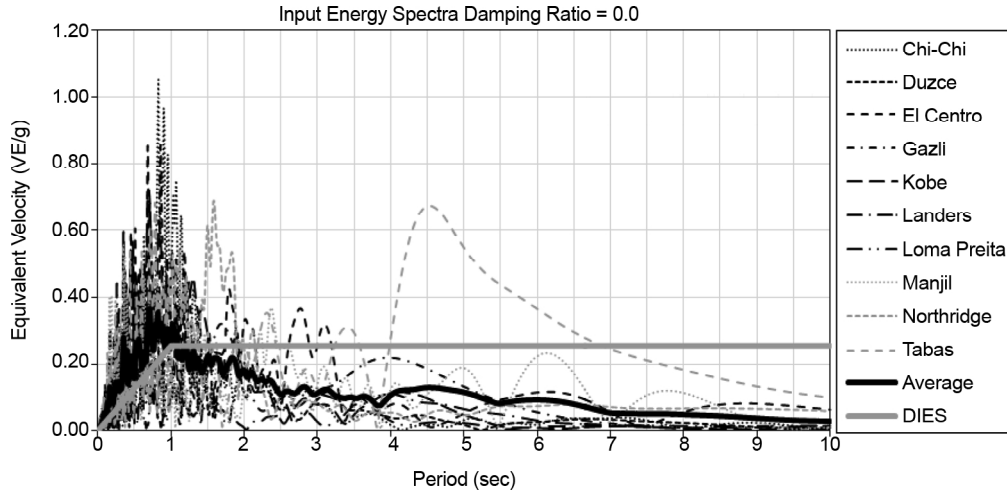


Figure 1. Input Energy spectra for $\zeta=0\%$.

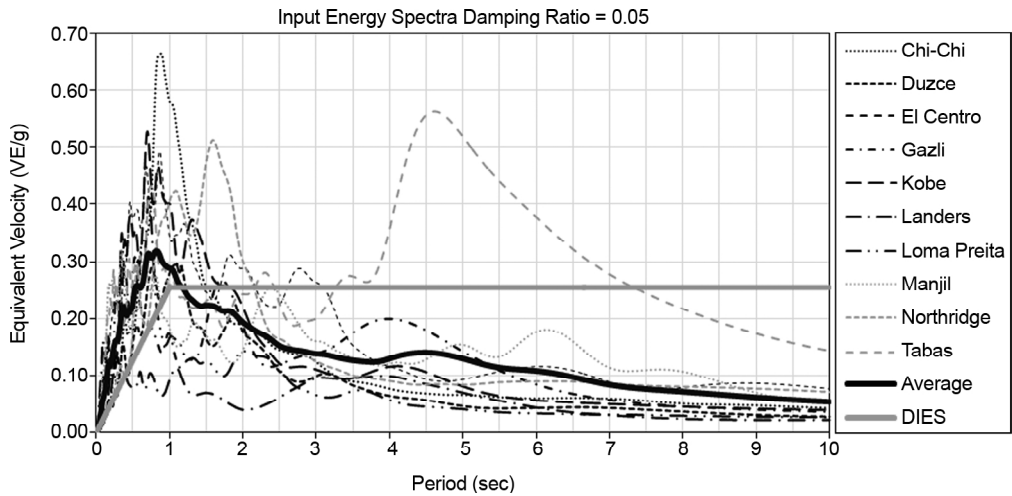


Figure 2. Input Energy spectra for $\zeta=0.5\%$.

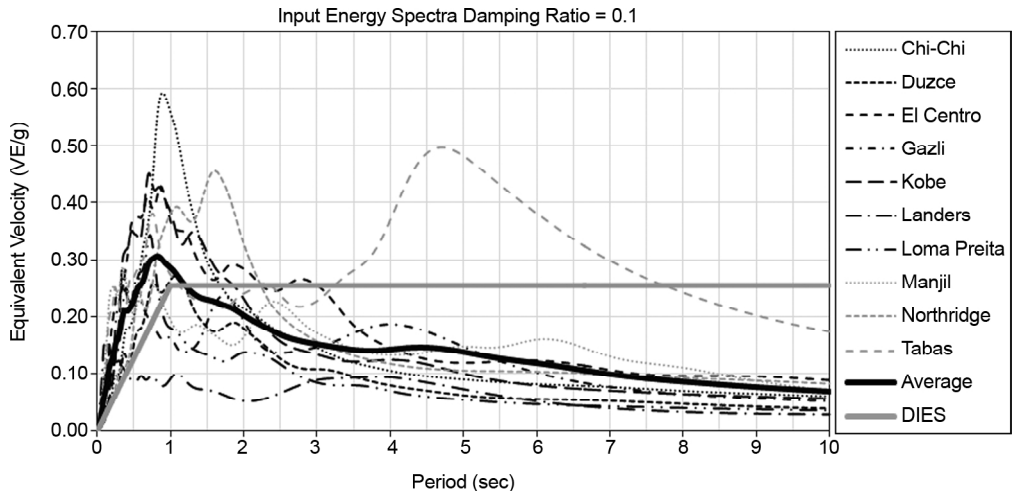


Figure 3. Input Energy spectra for $\zeta=10\%$.

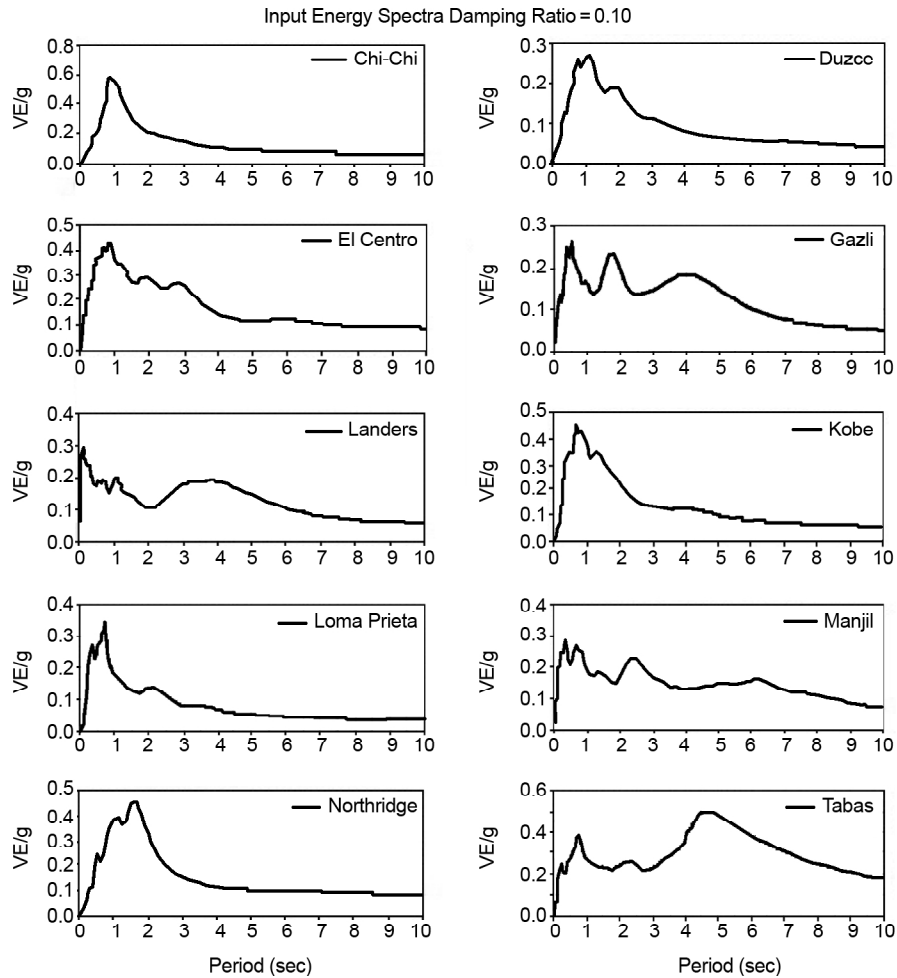


Figure 4. Input energy spectra of the considered earthquakes for $\zeta=10\%$.

Two important conclusions can be taken from these figures. First, the low decay rates of the average spectrum with increase in the period of vibration, especially in the practical range of periods of high-rise buildings, nearly 0.8 to 5.0 seconds. For example, it can be seen in Figure (3) that the amount of input energy at $T=5.0$ sec is half of that at $T=0.9$; however, typically, spectral pseudo acceleration at $T=5.0$ sec is less than one-fifth of the corresponding value at $T=0.9$ (considering *UBC97* design spectrum). The second important observation is the highly low sensitivity of the input energy to the damping of structures. Thus, the *DIES* can be assumed to be essentially constant over a wide range of periods, which is in agreement with Housner's pioneering statements. However, it is important to note that the issue is examined from the design spectrum point of view. This means that the results are valid for average values obtained from former earthquakes, not for an individual record. In addition, it should be noted that in order to obtain the input energy spectra

in this study, records were selected regardless of their site specification features, which does not affect the outcome of the study. However, several discrepancies can be observed between *DIES* and the average spectra as mentioned by other researchers [9].

It is worthy to know that most of the newly proposed elastic input energy spectra by researchers have the same spectral shape and character as illustrated by the average spectrum in Figures (1) to (3) [9-12].

3. Equation of Motion and Input Energy to Multi-Story Buildings

In the present study, a schematic illustration of the simplified model of multi-story buildings is demonstrated in Figure (5).

The equation of motion of the system in Figure (5) can be written as:

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = -\ddot{y}_g[M]\{r\} \quad (4)$$

or:

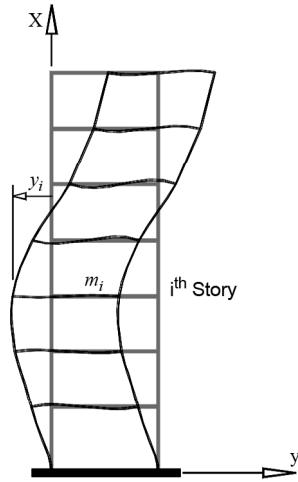


Figure 5. The structural model used in this study.

$$[M][\Phi]\{\ddot{z}\} + [C][\Phi]\{\dot{z}\} + [K][\Phi]\{z\} = -\ddot{y}_g[M]\{r\} \quad (5)$$

where:

$$\{y\} = [\Phi]\{z\}$$

$[\Phi]^T[M][\Phi] = [I]$ ($[\Phi]$ is orthonormalized and $[I]$ is the unit matrix.)

$[\Phi]^T[C][\Phi] = [c]$ ($[c]$ is a diagonal matrix with elements $c_{ii} = 2\zeta_i\omega_i \approx const.$)

$$[\Phi]^T[K][\Phi] = [\omega^2]$$

$$\{r\} = \{1\}$$

It is evident that the input energy to this system can be written as:

$$\begin{aligned} E &= -\int_0^{y_0} \ddot{y}_g \{d_y\}^T [M]\{r\} = \\ &= -\int_0^{t_d} \ddot{y}_g \{\dot{y}\}^T [M]\{r\} dt = \\ &= -\int_0^{t_d} \ddot{y}_g ([\Phi]\{\dot{z}\})^T [M]\{r\} dt \end{aligned} \quad (6)$$

Thus:

$$\begin{aligned} E &= -\int_0^{t_d} \ddot{y}_g \{\dot{z}\}^T [\Phi]^T [M]\{r\} dt = \\ &= \sum_{i=1}^n \left(-\int_0^{t_d} \ddot{y}_g \dot{z}_i \{\phi_i\}^T [M]\{r\} dt \right) \end{aligned} \quad (7)$$

Now, decoupling Eq. (5) results in:

$$\begin{aligned} [\Phi]^T [M][\Phi]\{\ddot{z}\} + [\Phi]^T [C][\Phi]\{\dot{z}\} + \\ [\Phi]^T [K][\Phi]\{z\} = \ddot{y}_g [\Phi]^T [M]\{r\} \end{aligned} \quad (8)$$

$$[I]\{\ddot{z}\} + [c]\{\dot{z}\} + [\omega^2]\{z\} = -\ddot{y}_g [\Phi]^T [M]\{r\} \quad (9)$$

$$\ddot{z}_i + 2\omega_i \zeta_i \dot{z}_i + \omega_i^2 z_i = -\ddot{y}_g \dot{z}_i \{\phi_i\}^T [M]\{r\} \quad (10)$$

Eq. (10) can be interpreted as equation of motion of a *SDOF* system with unit mass subjected to ground acceleration \ddot{y}_g , magnified by $\{\phi_i\}^T [M]\{r\}$. It is evident that magnifying the excitation by $\{\phi_i\}^T [M]\{r\}$ leads to magnification of the input energy by $(\{\phi_i\}^T [M]\{r\})^2$, thus considering Eq. (2) the input energy to the system presented by Eq. (10) can be written as:

$$\begin{aligned} -\int_0^{t_d} \ddot{y}_g \dot{z}_i \{\phi_i\}^T [M]\{r\} dt = \\ E_i (1 - \omega_i - \zeta_i) (\{\phi_i\}^T [M]\{r\})^2 \end{aligned} \quad (11)$$

By comparing Eqs. (7) and (11) the following equation can be written as:

$$E = \sum_{i=1}^n E_i (1 - \omega_i - \zeta_i) (\{\phi_i\}^T [M]\{r\})^2 \quad (12)$$

However, as indicated previously, in a wide range of relatively long to very long periods $E_i (1 - \omega_i - \zeta_i)$ has no notable variation and can be taken as constant, so Eq. (12) can be written as:

$$E = E_1 (1 - \omega_i - \zeta_i) \sum_{i=1}^n (\{\phi_i\}^T [M]\{r\})^2 \quad (13)$$

Based on Eq. (13), it can be claimed that E is considered as a constant value and the summation term in this equation should be constant as well. In fact, it can be stated that as the mode shapes of any system are “bases” of a “vector space” V , and each vector, including $\{1\}$ in this space can be written as a linear combination of “bases”, which is presented as:

$$\forall \{v\} \in V \exists a_i \in R \sum a_i \{\phi_i\} = [\Phi]\{a\} = \{v\}$$

Thus, if $\{v\} = \{1\}$ then $[\Phi]\{a\} = \{1\}$ and one can write:

$$[\Phi]^T [M][\Phi]\{a\} = [\Phi]^T [M]\{1\} \quad (14)$$

$$a_i = \{\phi_i\}^T [M]\{1\} \quad \text{and} \quad \{a\} = [\Phi]^T [M]\{1\} \quad (15)$$

Now the following equation can be written as:

$$\begin{aligned} \{a\}^T \{a\} &= ([\Phi]^T [M]\{1\})^T ([\Phi]^T [M]\{1\}) = \\ &= \{1\}^T [M]^T [\Phi][\Phi]^T [M]\{1\} \end{aligned} \quad (16)$$

However, $[\Phi][\Phi]^T [M]$ in the right hand side

of Eq. (16) should be a unit matrix because $[\Phi]^T[M][\Phi]=[I]$, and by pre-multiplication of both sides by $[\Phi]$, one can obtain the desired result. Thus, Eq. (16) can be rewritten as:

$$\{a\}^T \{a\} = \sum_{i=1}^n a_i^2 = \{1\}^T [M] \{1\} = \sum_{i=1}^n m_{ii} = m_{total} \quad (17)$$

total mass of MDOF structure

And finally, Eq. (13) can be written as:

$$E = m_{total} E_1(1, \omega_1, \xi_1) = E_1(m_{total}, \omega_1, \xi_1) \quad (18)$$

This means that the seismic input energy to a MDOF system is the same as input energy to a SDOF system with the same mass, main frequency and damping, provided that the following conditions are met:

- 1) Input energy is calculated at a specified instant of a record for all modes, nearly at the end of record.
- 2) Input energy spectra are constant all over the wide range of periods.

As it can be evident from Figures (1) to (3), constancy of input energy spectra is a simplifying assumption which apparently is not fully in agreement with reality. In fact, the input energy as expressed by Eq. (12) depends on the shape of the spectrum, and thus on the structural features. On the other hand, calculations of the input energy at a specified instant of a record, nearly at its end, imply that the input energy is the sum of instantaneous quantity in all modes. This fact simplifies the problem, and in conjugation with the inconstant input energy spectrum, it demonstrates the possibility of existence of an optimal distribution of stiffness along the height of high-rise buildings to minimize their seismic input energy.

4. Optimal Distribution of Stiffness

As shown, input energy to a MDOF system is a function of spectrum and structural properties. If mass distribution and damping of the system are assumed to be constant, then distribution of stiffness will be the unique effective structural property of linear MDOF systems. For shear structures, the distribution of stiffness can be defined by a $1 \times N$ vector as $k = \{k_i\}$, and for any given input energy spectrum, the seismic input energy to a MDOF system (E_n) will be merely a function of the stiffness distribution vector:

$$E_n = E_n(k) \quad (19)$$

The problem now is to find the optimum distribution of stiffness so that its value becomes minimal. The main constrain for the stories' stiffness values is obtained from the maximum acceptable story drift. Knowing the response spectrum, mass and damping of the system, maximum drift is merely a function of stiffness distribution:

$$maxDrift = f(k) \quad (20)$$

In the context of optimization problem, E_n is the objective function, k is design variable and $maxDrift = f(k) \leq d_{all}$ is inequality constraint (d_{all} is the maximum allowable story drift). The employed approach to solve this optimization problem is accomplished by applying a program developed in the MATLAB environment [13] which is illustrated in Figure (6).

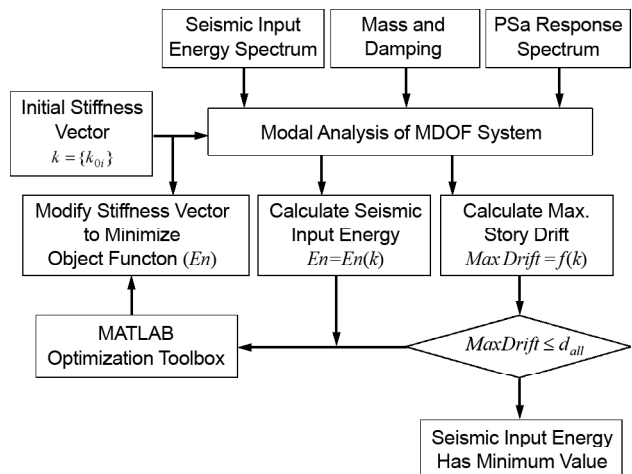


Figure 6. Optimization flowchart.

Various structures with different numbers of degrees of freedom were studied based on the aforementioned procedure [14]. In all cases, especially in structures with high degrees of freedom, there were different solutions to the problem, all with nearly the same value for minimum seismic input energy. However, in all studied structures, an interesting solution showed up. Based on the solution, the stiffness distribution should be in such a way that all stories reach the maximum allowable drifts limit; hence, it makes the modal displacement of the structure a linear form. This distribution for shear structures with low to moderate height (up to 20 stories), in which the first mode governs their

modal behavior, is best fitted by a parabolic curve, see Figure (7):

$$k(x) = k_0 / 2 - \alpha x^2 \tag{21}$$

For high-rise shear buildings (more than 20 stories), the optimal stiffness distribution looks similar to a bell-shaped curve and is best fitted by this curve, see Figure (8):

$$x = \alpha + \beta e^{-(k-\gamma)^2/\lambda} \Rightarrow k(x) = \gamma + \left[-\lambda \cdot \text{Ln} \left(\frac{x-\alpha}{\beta} \right) \right]^{1/2} \tag{22}$$

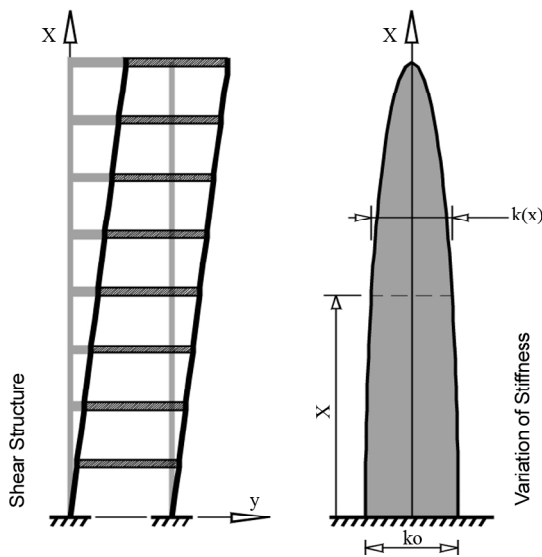


Figure 7. Optimal stiffness distribution for shear structures with low to moderate height (less than 20 stories).

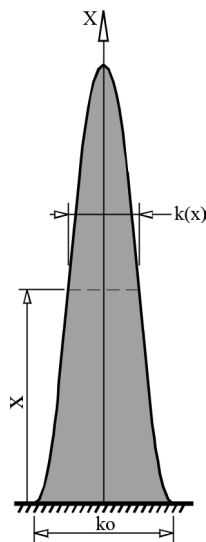


Figure 8. Optimal stiffness distribution for high-rise (more than 20 stories) shear structures.

In Eq. (21) k_0 is stiffness of the structure in its lowest story, and in Eq. (22) $\alpha, \beta, \gamma, \lambda$ are constant factors, related to the mass of system, input spectrum and accuracy of fit.

It is worth mentioning that based on the obtained results, the absolute minimum energy is achieved when the structure has the highest possible fundamental period; however, this is not practically possible due to drift limitation.

5. Conclusions

The following outcomes are the main conclusions of this study:

- ❖ The input energy spectrum is ascending in the range of short periods, and descending in the range of long periods, and therefore, the equality of the amount of input energy of a *MDOF* system, with that of a *SDOF* system (Housner's statement) is not valid in general. However, if the spectral value of input energy is assumed to be constant in the whole frequency range, then the Housner's statement will be valid.
- ❖ By selecting a specific distribution of stiffness along the height of multi-story shear buildings, from the spectral stand point, it is possible to minimize the amount of seismic input energy.
- ❖ For low to moderate height (up to 20 stories) shear buildings, one possible optimal distribution over the height is parabolic. For taller buildings, this form is a bell-shaped function. These distributions imply linearity of modal displacement of the building structure and the equality of drifts in all stories.

It should be noted that the *P-Delta* effect is omitted in this study; therefore, further research is required to include this effect. Finally, it should be pointed out that the obtained results are restricted to linear shear buildings, and more general nonlinear systems will be discussed as the second part of the study.

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