Research Note

Dynamic Analysis of Bilinear Oscillators Excited by Band-Limited White Noise Excitation

Mohammad-Rashid Salimi[1] and Azad Yazdani[2*]

1. Ph.D. Candidate, Department of Civil Engineering, University of Kurdistan, Sanandaj, Iran
2. Associate Prof., Department of Civil Engineering, University of Kurdistan, Sanandaj, Iran,
* Corresponding Author; email: a.yazdani@uok.ac.ir

Received: 03/02/2015
Accepted: 16/12/2015

ABSTRACT

The response spectrum of an oscillator with bilinear stiffness excited by band-limited Gaussian white noise is considered. The response is obtained by integrating over all energy levels weighting each with the stationary probability density of the energy. The procedure presented leads to estimates of linear and nonlinear response spectra in frequency domain and agrees well with those obtained by direct numerical simulation. Development of stochastic-based response spectra based on the frequency information concerning ground motions is important in engineering. Approximation of non-stationary ground motions by band-limited white noise is shown to be adequate for systems at the structural periods of engineering interest. Formulating the nonlinear response based on the excitation frequency information opens a door for wider use of seismological theory for regions with scarcely available recorded ground motion data. Despite simplicity and computational efficiency of the method, it provides an accurate prediction of the observed nonlinear response spectra on average.

Keywords:
Response; Band-limited white noise; Bilinear; Energy balancing

1. Introduction

In structural engineering, problems involving unpredictable variables or stochastic processes are frequently encountered. In these cases, a probabilistic analysis may be the most rational way of approaching the problem [1]. One of the main sources of uncertainty in the assessment of the actual response of structures is input ground motion excitation. In engineering applications, structural systems often display strong nonlinear behavior. Regarding the structural system, nonlinearities stem from geometry and/or material properties. However, probabilistic solution for nonlinear systems is difficult to obtain. In structural analysis, the stochastic seismic response analysis is rarely known, except for simple systems under idealized excitations [2]. With certain restrictive conditions, the state space vector is a Markov process, and the state space variable can be obtained as the solution to the corresponding Fokker-Planck equation [1]. Approximate solution techniques such as perturbation method [3], equivalent linearization method [4], equivalent non-linearization method [5], energy balance method [6], and numerical simulation method [7] are typically needed because such restrictive conditions can rarely be met in practical cases.

Methods for the evaluation of the response of linear and non-linear systems under Gaussian and non-Gaussian processes have been recently proposed by several authors, some of which are based on the theory of Markov processes and determine the probability density function of the response process [8]. Using the concept of equivalent linear system with random coefficients, Miles [9] estimated the power spectra density (PSD) response of a Duffing oscillator. Bellizzi and Bouce [10] presented a method to evaluate the higher harmonics of the PSD response of weakly damped oscillator with a nonlinear asymmetrical restoring force under external stochastic wide-band excitation. In a theoretical point of view, Lin and Cai [11, 12] and Cai and Lin [13] presented method based and an exact stationery solution for a class of systems termed the class of stationery potential. They indicated that approximate solutions can be obtained by replacing a stochastic system by an equivalent system belonging to the class of stationery potential for which solution exists. Furthermore, Lin and Cai [12] by using the concept of dissipation energy balancing and equivalent nonlinear system, replaced a nonlinear system by another nonlinear system so that the average dissipated energy in the two systems remains the same. This procedure is refined by Yazdani and salimi [14] to estimate spectral displacement of a hysteretic oscillator with bilinear stiffness excited by band-limited excitation. Rudinger and Krenk [1] considered a method to evaluate PSD of an oscillator with bilinear stiffness based on energy balancing procedure. Yazdani and Komachi [15] and Yazdani and Takada [16] respectively calculated the stochastic response of linear and Duffing oscillator under excitation based on the frequency information of ground motion. Zhu et al [17] obtained the stationary response stability and bifurcation of strongly nonlinear systems using the stochastic averaging technique for non-white, but not-so-narrow-band excitation. Kumar and Datta [18] presented a generalized procedure using the sto-chastic averaging technique for determining the response of strongly nonlinear single degree of freedom (SDOF) systems for cases where closed form solutions are not possible. Xiong et al [19] evaluated the dynamic behavior of system and the response regimes nonlinear mechanical system coupled to nonlinear energy sink based on the complex-averaging method and frequency detuning methodology under the impact of the narrow band stochastic excitation.

The purpose of this study is to estimate elastic and inelastic displacement and acceleration spectra of an SDOF with bilinear stiffness on the basis of generated Fourier amplitude spectra (FAS) of ground-motion using information on the seismic source, seismic wave propagation through the earth, and geological site conditions that affect ground-motion. The presented procedure based on stochastic techniques in frequency domain can be applied to regions where strong ground motion data are limited in availability regarding the magnitude and distance range of engineering interest. The procedure is validated by comparison with the results of conventional time domain response analysis for different recorded earthquakes.

2. Theoretical Analysis

When a nonlinear system is subjected to additive excitations of Gaussian white noise, the reduced Fokker-Planck equation can be solved in closed form only with certain highly restrictive relations between the system parameters and the spectral densities of the excitations. Rarely can such restrictive requirements be met in practical cases and for this reason the approximate solution techniques are generally necessary. The dynamic equilibrium equation for a SDOF system with linear damping and bilinear stiffness subjected to Gaussian white noise excitation $W(t)$ can be written as:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 g(x) = W(t) \tag{1}$$

where $x(t)$ is the displacement and dot indicates the derivative with respect to time. ω_n and ζ are the natural circular frequency and damping ratio in linear range, respectively. The nonlinear function $g(x)$ describes the bilinear force with pre-yielding slope of k and a post-yielding slope equal to αk. The yielding and maximum displacements are represented by x_y and x_m, respectively. The ratio x_m/x_y defines the ductility μ, which is a measure of nonlinearity degree in response. These characteristics of the force-displacement diagram are shown in Figure (1). Caughey [20] by assuming that the damping is only a function of the energy, using Fokker-Planck equation calculated the stochastic joint probability density of plane variable of displacement and velocity $(x, \dot{x})$ as:

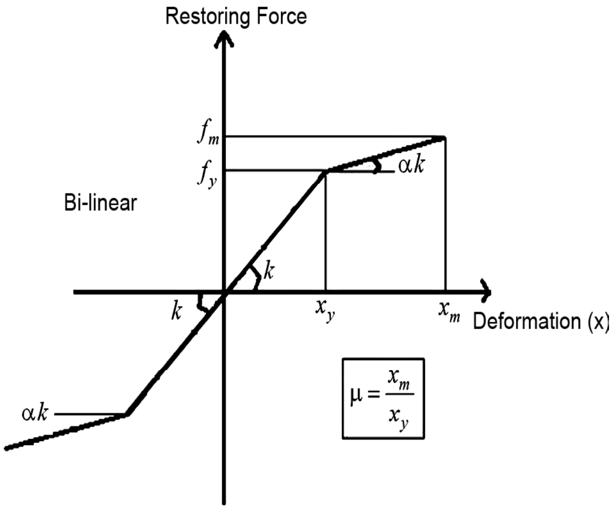

Figure 1. Force-deformation relationship of bilinear system.

$$P_{x,\dot{x}}(x,\dot{x}) = C\exp(-{2m\omega_n\xi\lambda}/{\pi S_0}) \tag{2}$$

where λ is the total mechanical energy that is equal to the sum of the kinetic energy and the potential energy. The potential energy is obtained by integration of the stiffness function. The constant C normalizes the density function of displacement and velocity. In this equation, the value of S_0 is equal to amplitude of the power spectral density function corresponding to natural frequency of system. Within the framework of the energy balancing, a set of weighting functions is selected, and the replacement system is required to satisfy the constraint that each weighted residual be zero. The main constraint is equivalent to dissipated energy balancing. By transformation from the plane variable $(x,\dot{x})$ to the energy and phase (λ,φ), the probabilit density of (λ,φ) can be defined as [21]:

$$P_{\lambda,\varphi}(\lambda,\varphi) = \frac{1}{(d\varphi/dt)}P_{x,\dot{x}}(x,\dot{x}) \tag{3}$$

The marginal probability density of the mechanical energy can be calculated by integrating over the phase as:

$$P_\lambda(\lambda) = T(\lambda)\exp(-{2m\omega_n\xi\lambda}/{\pi S_0}) \tag{4}$$

where $T(\lambda)$ is the period of free undamped vibration at energy level λ. In this case, the natural period is determined as [1]:

$$T(\lambda) = \begin{cases} \dfrac{2\pi}{\omega_n} & ;\lambda \le \dfrac{1}{2}\omega_n^2 x_y^{\,2} \\ \dfrac{4}{\omega_n}\sin^{-1}(\dfrac{\omega_n x_y}{\sqrt{2\lambda}}) + \dfrac{4}{\alpha\omega_n}\cos^{-1}\times & \\ (\dfrac{\omega_n x_y}{\sqrt{(\omega_1 x_y)^2 - (x_y\alpha\omega_n)^2 + 2\lambda\alpha^2}}) & ;\lambda > \dfrac{1}{2}\omega_n^2 x_y^{\,2} \end{cases} \tag{5}$$

In a system with bilinear stiffness, the probability density of energy at each energy level can be obtained. The ductility ratio of the system may be computed as follows:

$$\mu = \frac{x_m}{x_y} = \begin{cases} \dfrac{\sqrt{2\lambda}}{a\omega_n}, & ;\ \lambda \le \dfrac{1}{2}\omega_n^2 x_y^{\,2} \\ 1 - \dfrac{1}{\alpha^2}) + & \\ \dfrac{1}{\alpha}\sqrt{\dfrac{1}{\alpha^2} - 1 + \dfrac{2\lambda}{\alpha^2 x_y^{\,2}\omega_{n1}^2}} & ;\ \lambda > \dfrac{1}{2}\omega_n^2 x_y^{\,2} \end{cases} \tag{6}$$

The power spectral density function of ground motion is defined by $|F(\omega)|^2/T_{gr}$ in which $F(\omega)$ is the FAS of ground motion acceleration at frequency of ω, and T_{gr} is the earthquake ground motion duration [22]. There is a vast amount of research aimed at predicting the amplitude of the Fourier spectra, especially in the field of engineering seismology. Brune [23] assumes that the far-field accelerations on an elastic half-space are band-limited, finite-duration Gaussian noise and that the source spectra are described by a single corner frequency model whose corner frequency depends on the earthquake's size. One of the essential characteristics of this method is that, it distills what is known about the various factors affecting ground motions into simple functional forms. The far-field Fourier amplitude spectrum, $F(\omega)$, that has been used in seismological models can be broken into contributions from the earthquake source model (point-source), the typical geometric, inelastic whole path and upper crust attenuation; and site amplification functions [24].

$$|F(\omega)| = \frac{R_P F_S P}{4\pi\rho\beta^3}\frac{1}{R}E(\omega)An(\omega)P(\omega)A(\omega) \tag{7}$$

where R is the distance, R_P is the wave radiation factor (taken here as 0.55), F_S is the free surface amplification factor (taken to be 2), and P is the factor that partitions the energy into orthogonal directions (taken to be $\frac{\sqrt{2}}{2}$). ρ is the density of the rock within the top 10 km of the Earth's crust (typically 2.8 ton/m^3), and β is the shear-wave velocity in the vicinity of the source [23]. $E(\omega)$ is Brune's source spectrum, which is given as follows:

$$E(\omega) = \frac{M_0\omega^2}{\left(1+(\frac{\omega}{\omega_c})^2\right)} \tag{8}$$

where M_0 is the seismic moment, and ω_c is the corner frequency, which is given as follows:

$$\omega_c = (2\pi)\times 4.9\times 10^6 \beta\left(\frac{\Delta\sigma}{M_0}\right)^{1/3} \tag{9}$$

where, in this equation, the stress drop $\Delta\sigma$ has units of bars, ω_c has units of Hz, β_s has units of km/s, and M_0 has units of dyne-cm. The seismic moment M_0 is often expressed in terms of the moment magnitude M_w, which is defined as follows [25]:

$$M_w = 0.67\log M_0 - 10.7 \tag{10}$$

The loss of energy along the wave's travel path is very complex. By definition, the $An(\omega)$ factor includes all of the losses that have not been accounted for by the geometrical attenuation factor and is defined by the exponent expression, which is given as follows [24]:

$$An(\omega) = \exp\left(\frac{-0.5\omega R}{\beta.Q_0(\frac{1}{2\pi}\omega)^n}\right) \tag{11}$$

where Q_0 and n are the regional dependent factors of the wave transmission quality factor Q, which is defined by the exponent expression. The attenuation (or diminution) operator $P(\omega)$ in Eq. (7) accounts for the path independent loss of high-frequencies in the ground motions.

$$P(\omega) = \exp\left(-\frac{\omega\kappa}{2}\right) \tag{12}$$

where κ is the attenuation parameter that accounts for the high-frequency cutoff [26]. The term site effect is generally used to refer to wave propagation in the immediate vicinity of the site and not to propagation effects, which refer to the complete path from the source to the receiver. The boundary between a site effect and a propagation effect is not always clear, but it is useful to discuss them separately. In Eq. (7), $A(\omega)$ is the upper crust amplification factor and is a function of the shear-wave velocity versus the depth. The quarter-wave-length method proposed by Boore and Joyner [27] is used to model the amplification factor of the site. The geometrical attenuation can be defined from the developed trilinear attenuation model in accordance with the regional crustal thickness [24]; here, it is assumed to be $1/R$ for simplicity.

By substituting the stochastic point-source shear wave spectrum model into Eq. (4), it is indicated that the probability density of energy can be calculated as:

$$P_\lambda(\lambda) = T(\lambda)\exp(-\frac{53\pi m\omega_n\xi\rho^2\beta^6R^2\lambda(1+(\omega/\omega_c)^2)^2}{\omega^4M_0^2A^2(\omega)}\times \exp(\omega\kappa+\frac{\omega R}{\beta Q_0(\omega/2\pi)^2})) \tag{13}$$

The probability density function of energy can be computed in frequency domain, based on seismological information. The presented relation in calculating the response is based on the well-known stochastic model for generating strong motion, customarily used to calculate design earthquakes in places where there is a lack of sufficient recorded data. The stochastic point source model, which was used here, is based on a static corner frequency. Despite some theoretical deficiencies, this model gives similar results as the dynamic corner frequency version for medium and far distances and for ground motion frequencies of most interest to engineers [28]. The expectation of the energy can be obtained by integration of the energy with respect to its probability measure as:

$$E[\lambda] = \int_{-\infty}^{+\infty}\lambda P_\lambda(\lambda)d\lambda \tag{14}$$

Moreover, the coefficient of ductility of SDOF system, μ, is obtained by substituting the expected value of the energy in Eq. (6).

3. Numerical Results and Validation

In order to verify the validity of the presented approach to estimate the maximum deformations of SDOF system by bilinear stiffening and linear damping ratio, the results are compared with the linear and nonlinear response of SDOF system in time domain. In the proposed formulation in the frequency domain, the response is computed only by the FAS information based on the seismological variables. In places with lack of recorded data, the seismological variables can be calculated based on the stochastic models. The validity of the proposed approach is investigated through comparisons with results for the recorded earthquakes. Three different recorded ground motions, i.e. San Fernando 1971, Loma Prieta 1989, and Superstition Hills 1987 are considered. The simulated and observed FAS in three different events are presented in Figure (2). The good agreements achieved between simulated and observed FAS confirm the reliability of the utilized procedure in simulating ground motions. The values of seismological variables of these events [29] are indicated in Table (1). Based on the seismological information, by using Eq. (7), the probability density of energy can be calculated. Figure (3) indicates the probability density of the total energy of bilinear system for different coefficients of ductility at structural period of 0.2 and 1.0 s. By calculating the level of energy, the coefficient of ductility based on seismological information in frequency domain is obtained.

Figures (4) and (5) compare the linear and nonlinear response spectrum obtained by the conventional time domain method and the proposed frequency domain approach with parameters α = 0.03, ζ = 0.05. In this figure, the solid line shows the response spectrum evaluated using the time domain approach and the dashed line illustrates response spectrum obtained in frequency domain. Different traces compare the displacement and

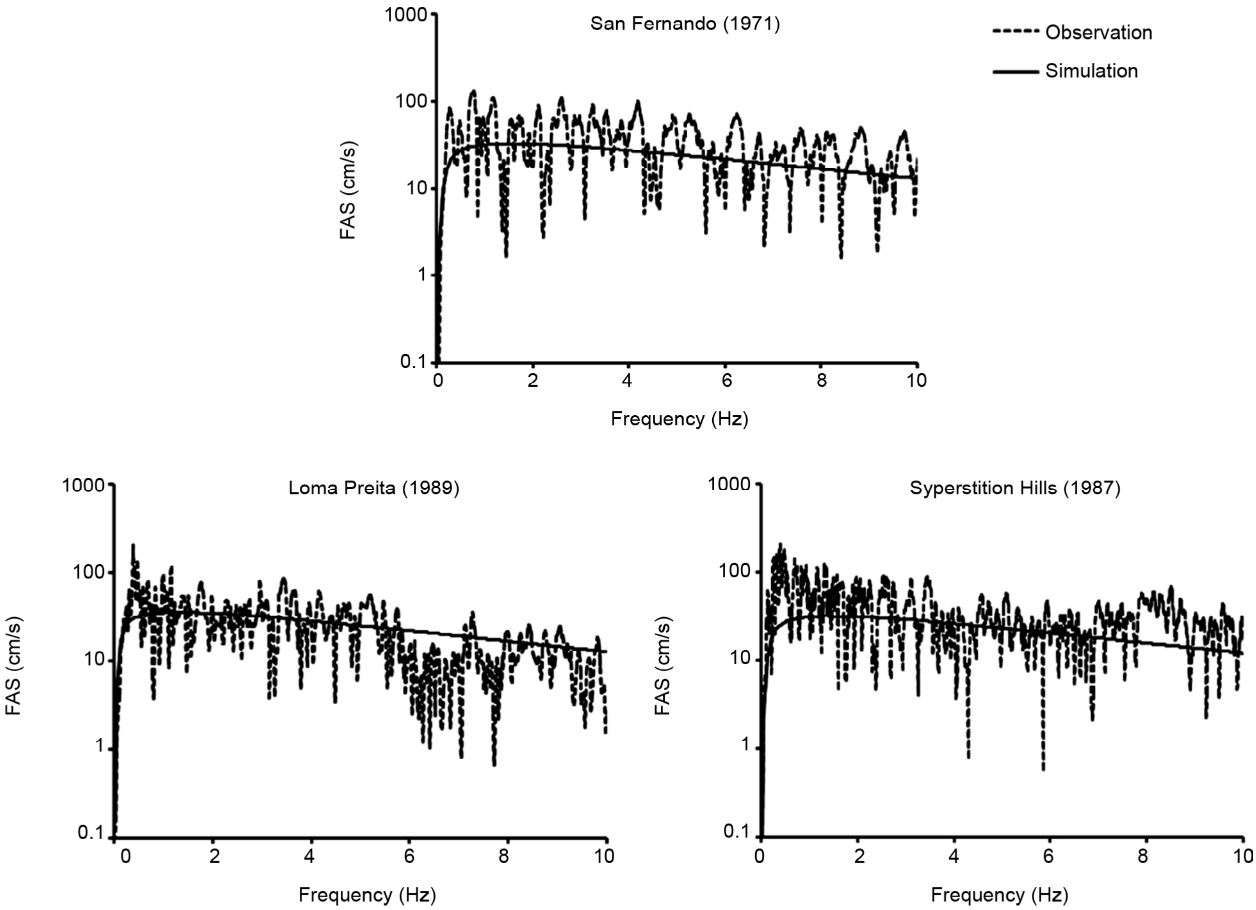

Figure 2. Comparison of the observed (dash line) and the simulated Fourier amplitude Spectra (solid line) for three different ground motions.

Table 1. Set of variables of used strong motions in simulation.

Random Variables	San Fernando (1971)	Loma Prieta (1989)	Superstition Hills (1987)
Earthquake Magnitude, Mw	6.6	6.9	6.7
Distance, R(km)	21.2	28.2	24.4
Density, $\rho_S(gr/cm^3)$	2.8	2.8	2.8
Shear-Wave Velocity, $\beta_S(km/s)$	3.7	3.7	3.7
Stress Drop, (bar)	80	80	80
Quality Factor, $Q(w)$	$180*f^{0.45}$	$180*f^{0.45}$	$180*f^{0.45}$
High-Frequency Attenuation Parameter, κ (s)	0.04	0.04	0.04
Duration, T_W(s) [30]	$2\pi/\omega_c + 0.05R$	$2\pi/\omega_c + 0.05R$	$2\pi/\omega_c + 0.05R$
Amplification Factor, NEHRP Class	C	C	D

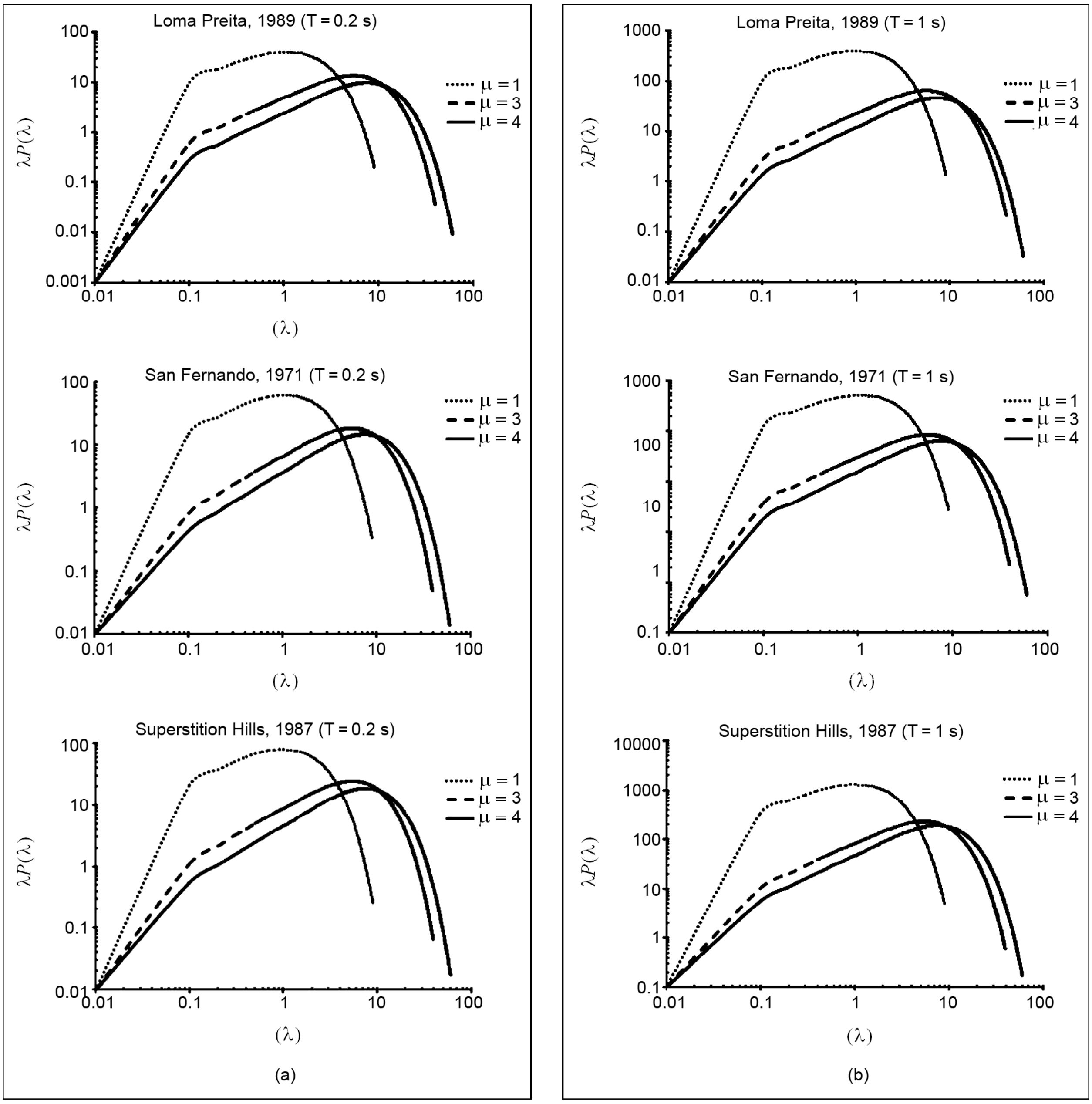

Figure 3. The probability density of the total energy of bilinear system with α = 0.03 for three different coefficients of ductility and damping ratio equal to 0.05 (a) at structural period of 0.2 s (b) at structural period of 1.0 s.

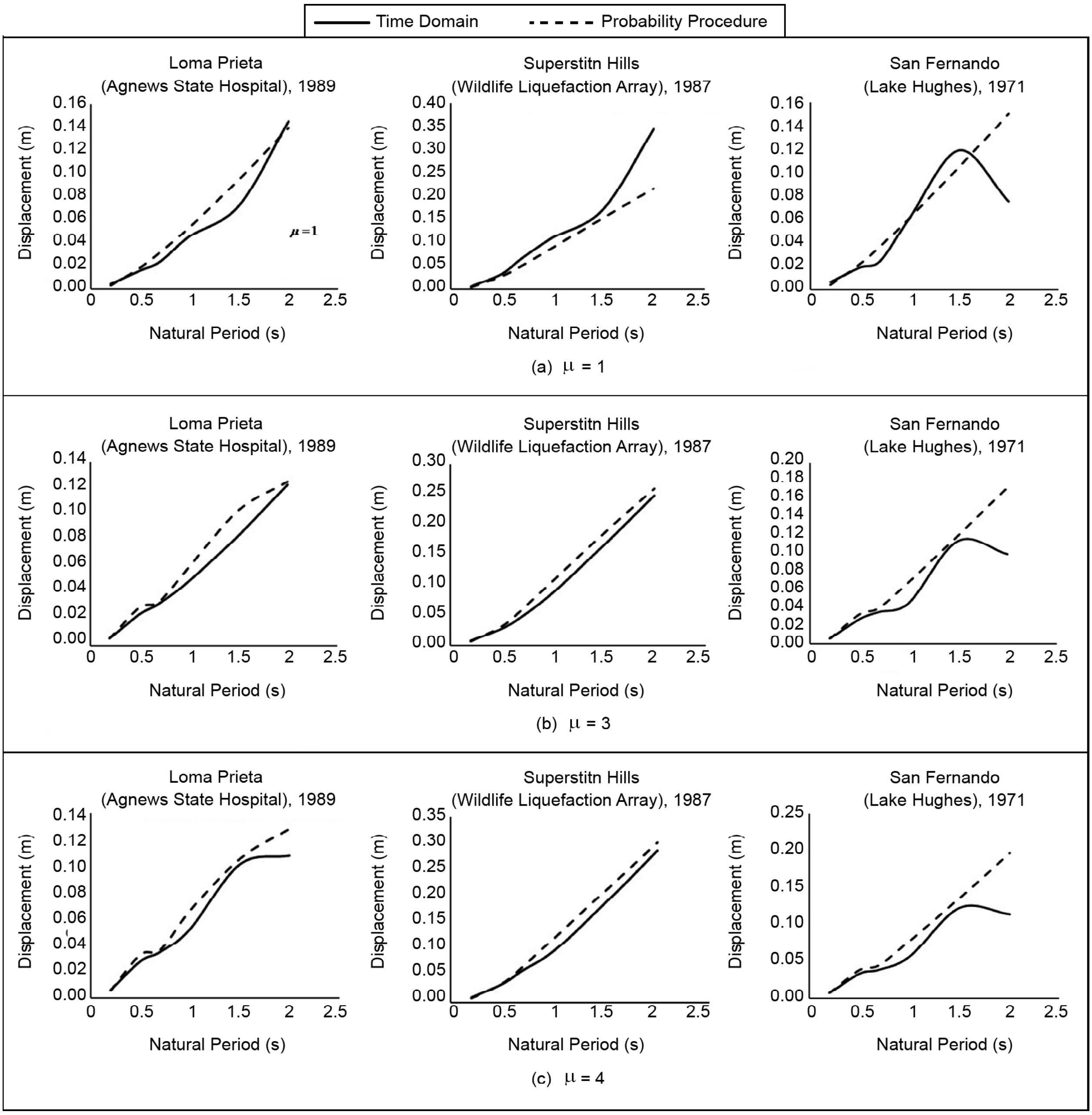

Figure 4. Comparison of the linear and nonlinear displacement response spectra in the proposed frequency domain (dash line) and in time domain approaches (solid line) when ζ=0.05 for three different ground motions.

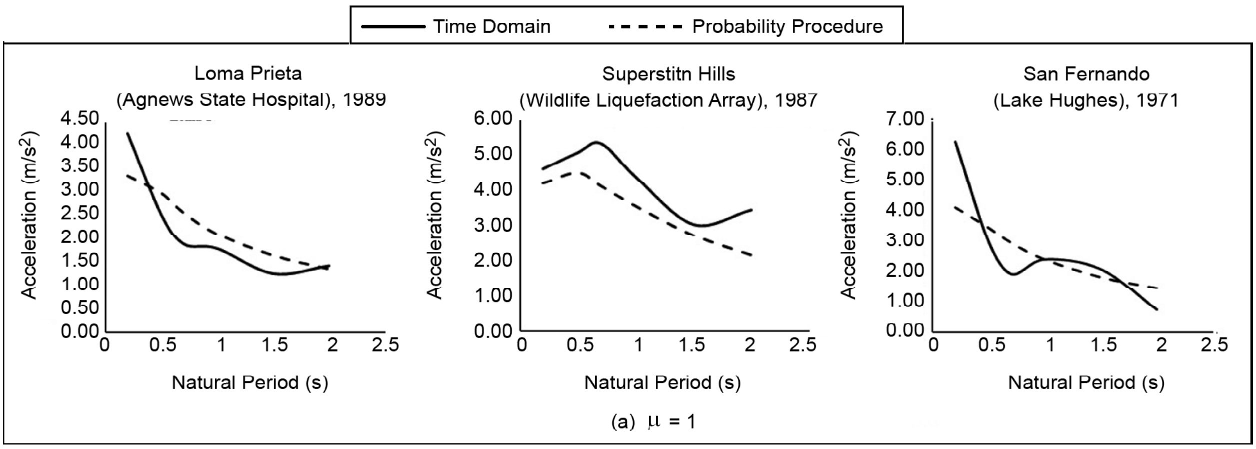

Figure 5. Comparison of the linear and nonlinear displacement response spectra in the proposed frequency domain (dash line) and in time domain approaches (solid line) when ζ=0.05 for three different ground motions.

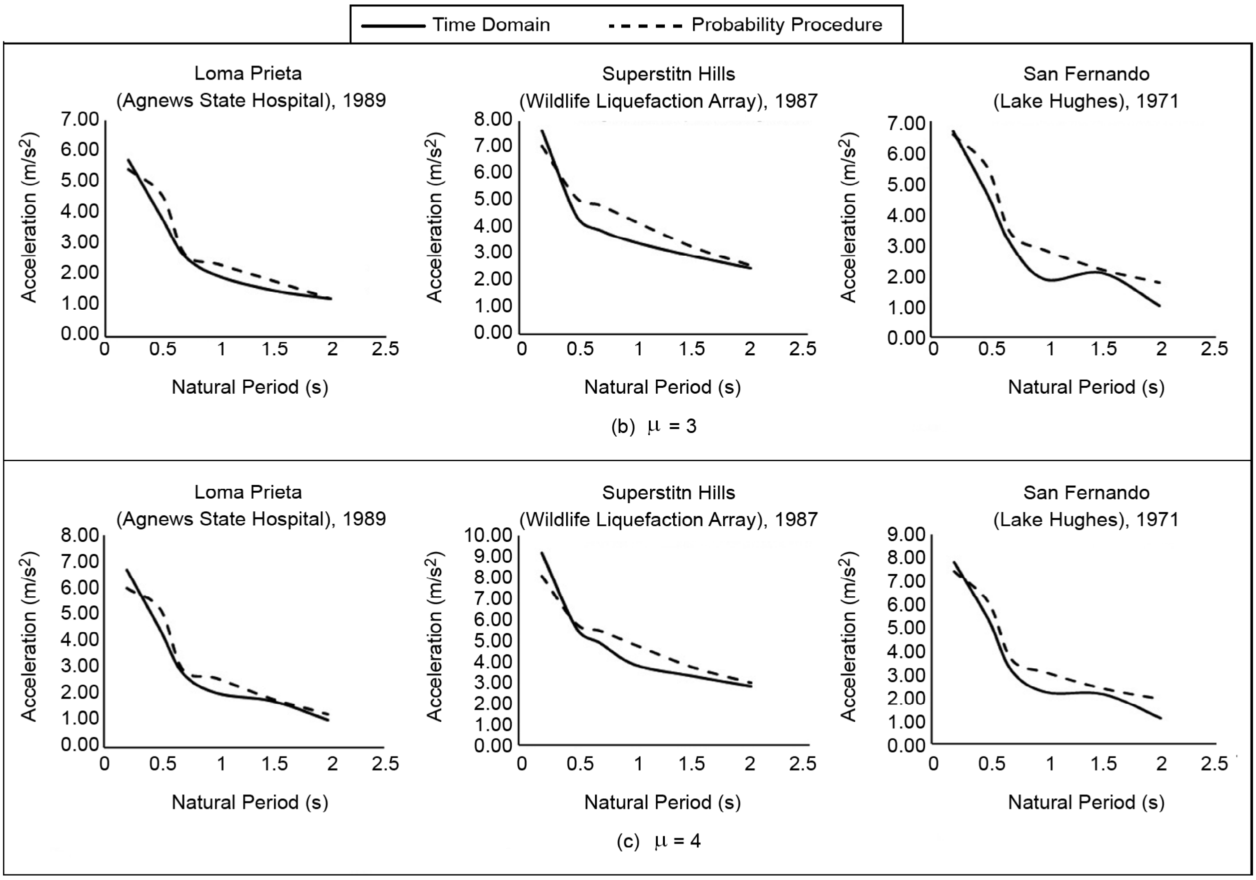

Figure 5. Continue.

acceleration spectra for different events for three different coefficients of ductility. Discrepancies of several percent can be found in the frequency domain and time domain methods. These discrepancies are caused by the difference in the procedure and the input seismological parameters. The acceptable mismatch between results based on seismological FAS and recorded information verify the integrity of the proposed approach.

4. Discussions and Conclusions

The formulation of the response spectrum in the frequency domain is appropriate for computing the response when the FAS information is available. The presented formulation of dynamic structural response in the frequency domain, based on only the frequency information of the excitation, provides an important basis for structural analysis in some countries, such as Iran, that lacks strong motion records for dynamic analysis. In the estimation of dynamic responses, a suite of ground motion time histories needs to be utilized in nonlinear time domain analysis. These ground motions are limited by the amount of available strong motion data and by the fact that they are based on combined recorded data sets from different earthquakes recorded in different regions. In some locations for which there is a lack of sufficient recorded data, well-known stochastic models are customarily used to simulate strong motions for the purpose of structural analysis. Some of the uncertainties in design of safer structures result from the lack of information due to the low occurrence rate of large earthquakes and this problem cannot be resolved in a practical time span. It is, therefore, strongly desirable to develop a robust method taking into account these uncertainties with limited information and enabling the design of safer structures.

One of the essential characteristics of the seismological method is that it distills what is known about the various factors affecting ground motions into different functional forms. The presented expression in this study provides an important basis for a wider use of seismological theory in structural analysis. For regions where recorded ground motion data are scarce, it becomes imperative to use the proposed models to represent the linear and nonlinear response of structures.

The accuracy of the proposed formulation was investigated by comparing the results with those obtained from conventional time domain. While neither the conventional time domain response spectra nor the recorded Fourier spectra are appropriate to use as a measure, because of the lack of information on recorded data, the proposed measure overcomes the problem. Despite possessing some deficiencies the presented processor, formulating a response spectrum on the basis of information on the frequency excitation opens the door for wider use of seismological theory in understanding the relationship between the linear and nonlinear response spectra and the seismological variables of interest. Due to the simplicity and computational efficiency of the method, it provides an accurate prediction of the observed nonlinear response spectra on average for structural periods of engineering interest.

Acknowledgments

The authors would like to thank anonymous reviewers for comments which helped to improve the manuscript.

References

1. Rudinger, F. and Krenk S. (2003) Spectral density of oscillator with bilinear stiffness and white noise excitation. *Probabilistic Engineering Mechanics*, **18**, 215-222.
2. Koliopulos, P.K. and Nichol, E.A. (1994) Comparative performance of equivalent linearization techniques for inelastic seismic design. *Engineering Structures*, **16**(1), 5-10
3. Crandall, S.H. (1963) Perturbation techniques for random vibration of nonlinear systems. *J. Acoust. Soc. Am.*, **35**, 1700-1705.
4. Caughey, T.K. (1971) *Nonlinear Theory of Random Vibrations, Advances in Applied Mechanics*. New York: Academic Press, 209-253.
5. Roberts, J.B. and Spanos, P.T.D. (1986) Stochastic averaging: An approximate method of solving random vibration problems. *Int. J. Nonlinear Mech.*, **21**, 111-134.
6. Dimentberg, M.F. (1996) Random vibrations of an isochronous SDOF bilinear system. *Nonlinear Dynamics*, **11**, 401-405.
7. Iourtchenko, D.V. and Song, L.L. (2006) Numerical investigation of a response probability density functions of stochastic vibroimpact systems with inelastic impacts. *International Journal of Non-Linear Mechanics*, **41**, 447-455.
8. Manohar, C.S. (1995) Methods of nonlinear random vibration analysis. *Sadhana*, **20**, 345-371.
9. Miles, R. (1989) An approximation solution for spectral response of Duffing's oscillator with random input. *Journal of Sound and Vibration*, **132**, 43-49.
10. Bellizzi, S. and Bouc, R. (1996) Spectral response of asymmetrical random oscillators. *Probabilistic Engineering Mechanics*, **11**(5), 1-9.
11. Lin, Y.K. and Ca, G.Q. (1988) Exact stationary response solution for second order nonlinear systems under parametric and external white noise excitations: Part II. *Journal of Applied Mechanics*, **55**, 702-705
12. Lin, Y.K. and Cai, G.Q. (2004) *Probabilistic Structural Dynamics: Advanced Theory and Applications*. McGraw-Hill, New York, 546p.
13. Cai, G.Q. and Lin, Y.K. (1997) Response spectral densities of strongly nonlinear systems under random excitation. *Probabilistic Engineering Mechanics*, **12**, 41-47.
14. Yazdani, A. and Salimi, M-R. (2015) Earthquake response spectra estimation of bilinear hysteretic systems using random-vibration theory method. *Earthquakes and Structures*, **8**(5), 1055-1067.
15. Yazdani, A. and Komachi, Y. (2009) Computation of earthquake response via fourier amplitude spectra. *International Journal of Engineering*, **22**(2), 147-152.
16. Yazdani, A. and Takada, T. (2011) Probabilistic study of the influence of ground motion variables on response spectra. *Structural Engineering and Mechanics*, **39**(6), 877-893.
17. Zhu, W.Q., Haung, Z.L., and Suzuki, Y. (2001) Response and stability of strongly nonlinear

oscillators under wide-band random excitation. *Int. J. Nonlinear Mech.*, **36**, 1235-1250.

18. Kumar, D. and Datta, T.K. (2006) Stochastic response of nonlinear system in probability domain. *Sadhana*, 31(4), 325-342.

19. Huai, X., Xianren, K., Zhenguo, Y., and Yuan, L. (2015) Response regimes of narrow-band stochastic excited linear oscillator coupled to nonlinear energy sink. *Chinese Journal of Aeronautics*, **28**(2), 457-468

20. Caughey, T.K. (1986) On the response of non-linear oscillators to stochastic excitation. *Probabilistic Engineering Mechanics*, **1**, 2-10.

21. Krenk, S. and Roberts, J.B. (1999) Local similarity in non-linear random vibration. *Journal of Applied Mechanics*, **66**, 225-235.

22. Takewaki, I. (2001) Resonance and criticality measure of ground motions via probabilistic critical excitation method. *Soil Dynamic and Earthquake Engineering*, **21**, 645-659.

23. Brune, J.N. (1970) Tectonic stress and the spectra of seismic shear waves from earthquakes. *Journal of Geophysical Research*, **75**, 4997-5009.

24. Boore, D.M. (2003) Prediction of ground motion using the stochastic method. Pure Appl. *Geophys.*, **160**, 35-676.

25. Kanamori, H. (1977) The energy release in great earthquakes. *J. Geophys Res.*, **82**, 2981-2987.

26. Anderson, J.G. and Hough, S. (1984) A model for the Fourier amplitude spectrum of acceleration at high frequency. *Bull. Seismol. Soc. Am.*, **74**(5), 1969-1993.

27. Boore, D.M. and Joyner, W.B. (1997) Site amplification for generic rock sites. *Bull. Seismol. Soc. Am.*, **87**(2), 327-341.

28. Boore, D.M. (2009) Comparing stochastic point-source and finite-source ground-motion simulations: SMSIM and EXSIM. *Bull. Seismol. Soc. Am.*, **99**(6), 3202-3216.

29. Atkinson, G.M. and Silva, W. (2000) Stochastic modeling of California ground motions. *Bull. Seismol. Soc. Am.*, **90**, 255-274.

30. Atkinson, G.M. and Boore, D.M. (1995) Ground motion relations for eastern North America. *Bull. Seismol. Soc. Am.*, **85**, 17-30.

Symbols

A: Upper crust amplification factor
C: Normalizing constant
F: Fourier amplitude spectrum
R: Source to site distance
T: Free vibration period
W: Unit white noise processes
M_0: Seismic moment
Q_0: Regional dependent factor of the wave transmission quality
S_0: Intensity of additive white noise process
T_{gr}: Earthquake ground motion duration
g: Stiffness function
m: Effective mass
x: Displacement
t: Time
x_y: Yielding displacement
x_m: Maximum displacement
ζ: Damping ratio
ω_n: Natural circular frequency
λ: Mechanical energy
φ: Phase
α: Post-yield hardening ratio
μ: Ductility
β: Shear-wave velocity
ρ: Density
κ: Attenuation parameter that accounts for the high-frequency cutoff
ω_c: Corner frequency
$E[.]$: Mean value operator
$P(.)$: Stochastic joint probability density

Abbreviations

FAS: Fourier Amplitude Spectrum
PDF: Probability Density Function
PSD: Power Spectral Density
SDOF: Single Degree Of Freedom

* Dots over variable denote derivatives with respect to time.