A New View on Optimal Control Algorithms

Rahman Mirzaei[1*] and Omid Bahar[2]

1. Ph.D. Candidate, School of Engineering, Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran, * Corresponding Author; email: rah.Mirzaei@gmail.com
2. Assistant Professor, Structural Dynamics Department, International Institute of Earthquake Engineering and Seismology (IIEES), Iran

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ABSTRACT

During past decades, many control algorithms with some advantages/weaknesses have been proposed. However, the most famous and historic algorithm which has found widespread applications in different fields of science, is the family of optimal control method (OCM). Today, this family includes many different approaches using various performance indices in continuous or discrete domain of consideration. The main stem of OCM is rooted in a simple definition, say performance index (PI), which should be minimized with respect to the main independent variables of the system. Although, different proposed algorithms are employed various performance indices besides simple or stable weighting matrices, their performances are noticeably similar. Extensive analysis shows that the main aspect, which results in reasonable similar performances in spite of their assumption for determining control force, arise from solving the Riccati matrix equation (RME) during their procedure or their stability criteria. This idea is examined by introducing a new simple but unusual assumption, named the simplified LQR (SLQR), via considering seismic behavior of an eight-story shear type building structure.

Keywords:
Optimal control method; Riccati matrix equation; Simplified LQR; SLQR; AMD

1. Introduction

Today, it is a common believe that active structural control systems regardless of their practical drawbacks may be a major part of new or existing structures in the near future. It is an excellent anti-seismic solution for designing and/or enhancing performance of structures against extraordinary environmental loads. Application of active structural control theory to the civil engineering structures is well documented in the literatures [1-2]. In control theory, the simple widespread algorithms of optimal control methods are now a large family, which are designed based on an optimizing procedure using different approaches like the calculus of variations, Pontryagin's maximum principle, Bellman dynamic programming, or Krotov method [3]. Among them, it can be mentioned to the classic optimal control theory [2-3], the instantaneous optimal control method [4-5], generalized optimal active control algorithm [6], the instantaneous optimal Wilson-θ control method [7], and the discrete instantaneous optimal control method [8]. These algorithms by minimizing a proper regulator or performance index defined based on the parameters of the structure or external loads provide control forces in order to achieve the best performance of the system in view of optimality. Among different proposed methods, the most famous and historic algorithm which has found widespread applications in different fields of science, is linear quadratic regulator, *LQR*. It is also superior, because of its simple procedure and ease of implementation on actual large-scale systems [4]. Only classical closed-loop control is applicable to the structural control problem. However, since the Riccati equation is obtained by ignoring the earthquake excitation term, classical closed-loop control is in fact approximately optimal and does not satisfy all optimality conditions. On the other hand, although the classical open-closed loop and open loop algorithms are superior to the closed loop algorithm, since they need prior knowledge of the entire external loading time history are not applicable [2].

To overcome shortcomings of the classical optimal control algorithms, several methods have been proposed. One of these algorithms, which has already attracted lots of attention is instantaneous optimal control method introduced by Yang et al [4-5] and later developed in a discrete form by Bahar et al [8-9]. This procedure has a main advantage that is on-line controlling of structures, especially against earthquake load excitations. However, this algorithm is very sensitive to time steps such that proper selecting of weighting matrices is very difficult [9].

Cheng and Tiang [6] developed a technique called the generalized optimal active control (*GOAC*) for seismic-resistant structures. They have defined a new performance index in discrete time domains, in which the control time interval is divided into infinitesimal distance as small as the time step size. Optimal control force is then achieved via minimization of the new defined performance index. Bahar et al [7] have introduced the instantaneous optimal Wilson-θ method in which dynamic equation of motion is discertized by means of the Wilson-θ method. In spite of suitable performance; however, the proposed algorithm, like other algorithms in this class, is sensitive to change of time increment. To suppress this deficiency, Yang et al [10] using the Lyapunov direct method proposed a stable weighting matrix that highly improves efficiency of the method.

In this paper, a new simplified algorithm based on conventional closed loop linear quadratic regulator but an unusual criterion is defined. The proposed method using an assumption presents an easy way to implement optimal algorithms in structural control. Gain matrix is computed directly from control force derivative; therefore, control force is obtained as fast as possible. This algorithm solves time increment problem, but in general, determined control force may not always guarantee the stability of the building and also high efficiency of the control system. Hence, stability of the system is achieved by means of the Lyapunov stability criteria, which results in a proper weighting matrix.

2. The Family of Optimal Control Methods

For a building equipped with an active control system subjected to a ground excitation, the linear equations of motion can be written in matrix form as:

$$M\ddot{x} + C\dot{x} + Kx = -ME\ddot{x}_g + Du(t) \tag{1}$$

where $x=\{x_1, x_2, \ldots, x_n\}^T$ is the n-dimension vector of the relative displacements, M, C and K are $n\times n$ matrices of, respectively, mass, damping and stiffness of the structure, and $\ddot{x}_g$ is the ordinates of a ground acceleration record. E is an $n\times 1$ influence vector of the ground forces, $D_{n\times m}$ is the location matrix of the control forces and $u(t)$ is an $m\times 1$ control force vector applied by m actuators. The equation of motion can be easily written in terms of the state space variable, Z, as follows:

$$\dot{Z} = AZ(t) + Bu(t) + Hf(t) \quad , \quad Z(t_0) = Z_0 \tag{2}$$

where t_0 is the initial time instant; vector $Z(t)$ is the state space vector; and matrix A is the system matrix, defined as follows, respectively:

$$Z(t) = [x(t), \dot{x}(t)]^T \text{ and } A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \tag{3}$$

Matrix B, and vector H are given as follows:

$$B = \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 \\ -E \end{bmatrix} \tag{4}$$

By designing a proper controller, control force, i.e. $u(t)$ in Eq. (1) or (2), is determined from the measured structural response and/or external loading. Various algorithms are proposed, but here, we only paid attention to the family of optimal control methods. In this family by introducing a performance index, it is tried to maximize the reduction of structural response with minimum control energy or control force consumption. Different algorithms have been used various performance indices, which leads to variety of control gains. In the following parts, the concept of a few familiar performance indices and determination of feedback gain and control force ones are presented in brief.

2.1. Linear Quadratic Regulator-Closed Loop (CCLQR)

In *LQR* method, the optimum control forces are determined via tuning some weighting matrices during the minimization of the performance index, J, defined by:

$$J = \int_0^{t_f} [Z^T(t)QZ(t) + u^T(t)Ru(t)]dt \tag{5}$$

in which Q is a $2n\times 2n$ positive semi-definite matrix, R is a $r\times r$ positive definite matrix and t_f is the terminal time that should be longer than earthquake duration [2]. To minimize the performance index J subjected to the constraint given by Eq. (2), the necessary conditions are as follows:

$$\dot{\lambda} = -A^T\lambda(t) - 2QZ(t) \tag{6}$$

$$u(t) = -\frac{1}{2}R^{-1}B^T\lambda(t) \tag{7}$$

in which $\lambda(t)$ is a $2n$ vector representing the Lagrange multiplier. The optimal control vector $u(t)$, the Lagrange vector $\lambda(t)$, and the state vector $Z(t)$ can be solved using Eqs. (2), (6) and (7). Notice that the control vector $u(t)$, in Eq. (7), is directly related to the Lagrange vector $\lambda(t)$. If the control force is assumed to be proportional to the state vector $Z(t)$, the *LQR* control is named optimal closed-loop, *CCLQR*, control. In this case, one has:

$$\lambda(t) = P(t)Z(t) \tag{8}$$

where $P(t)$, called the Riccati matrix, is obtained by solving the following nonlinear matrix equation:

$$\begin{aligned}&\left[\dot{P}(t)+P(t)A-\frac{1}{2}P(t)BR^{-1}B^TP(t)+A^TP(t)+2Q\right]\times\\&Z(t)+P(t)Hf(t)=0,\ P(t_f)=0\end{aligned} \tag{9}$$

There are two assumptions to solve Eq. (9); (1) the external excitation, $f(t)$, is equal to zero or it is a white noise stochastic process, (2) the Riccati matrix, $P(t)$, in most cases can be approximated by a time-invariant matrix P. Hence, Eq. (9) reduces to the following equation:

$$\begin{aligned}&P(t)A-\frac{1}{2}P(t)BR^{-1}B^TP(t)+A^TP(t)+2Q=0\\&P(t_f)=0\end{aligned} \tag{10}$$

Upon this equation, a constant control gain matrix is attained as follows:

$$G = -\frac{1}{2}R^{-1}B^TP \tag{11}$$

Choosing proper Q and R weighting matrices, and solving the matrix Riccati equation, Eq. (10), a specific semi-optimal control gain matrix, G, is obtained.

2.2. Instantaneous Optimal Control (IOC)

Yang et al [5] have developed Instantaneous optimal control (*IOC*) algorithm. The basic idea of the derivation of the instantaneous optimal control is relied on assuming a time dependent quadratic performance index without integration over the time, as follows:

$$J(t) = Z^T(t)QZ(t) + u^T(t)Ru(t) \tag{12}$$

where Q and R are before mentioned weighting matrices. If the direct solution of Eq. (2) is considered and trapezoidal rule is employed to compute the consequent integration, we obtain [6, 8]:

$$\begin{aligned}&Z(t)=exp(A\Delta t)\times\\&\left\{Z(t-\Delta t)+\frac{\Delta t}{2}\left[Bu(t-\Delta t)+H\ddot{x}_g(t-\Delta t)\right]\right\}+\\&\frac{\Delta t}{2}\left[Bu(t)+H\ddot{x}_g(t)\right]\end{aligned} \tag{13}$$

By minimizing performance index $J(t)$ subjected to the constraint given in Eq. (13), the necessary conditions are attained as follows:

$$\lambda(t) + 2QZ(t) = 0 \tag{14}$$

$$u(t) = -\frac{\Delta t}{2}R^{-1}B^TZ(t) \tag{15}$$

where λ is a Lagrange multiplier vector. Based on Eq. (15), the active control force is directly proportional to the time increment λt, which may cause decreasing in control efficiency by using smaller time increment [7, 9]. To overcome this shortcoming and also guarantee the stability of the controlled structure, Yang et al [10] have proposed a method to determine stable weighting matrices using the Lyapunov stability theory. In this method, by choosing a proper positive semi-definite matrix, I_0, and solving the following Riccati matrix equation, the appropriate continuous Q weighting matrix is determined:

$$A^TQ - \Delta tQBR^{-1}B^TQ + QA^T + I_0 = 0 \tag{16}$$

2.3. Generalized Optimal Active Control Algorithm (GOAC)

Cheng and Tian [6] developed a technique called

GOAC algorithm for seismic-resistant structures. They have defined a new performance index in discrete time domain, in which the control time interval is divided into *N* segments similar to time step size. Optimal control force in this case is achieved via minimization of the new defined performance index:

$$J = \sum_{i=1}^{N} \frac{1}{2} \int_{t_{i-1}}^{t_i} \left[Z^T Q Z + u^T R u \right] dt \tag{17}$$

This performance index is integrated step by step in discrete time domain. In each step, the end of the domain is assumed not to be fixed, which leads to a variational problem with transversality condition at the end point. This procedure yields to the following feedback gain matrix:

$$G = R^{-1} B^T S \tag{18}$$

and the control force is defined as:

$$u(t) = R^{-1} B^T S Z(t) \tag{19}$$

The gain matrix is invariant with respect to time and it is valid at every end point. It is clear that if *S* matrix is chosen from the algebraic Riccati matrix equation like Eq. (10), the algorithm will be the same as the classical optimal control, closed-loop control. However, in general, there is no any suggestion to select a proper *S* such that stability of the whole building during control time is guaranteed and also the control system acts with high efficiency.

2.4. Instantaneous Optimal Wilson-θ Control Method

To obtain a stable control method against earthquake ground motions, it seems using an implicit numerical method is a very useful tool for discretization of the dynamic equation of motion, Eq. (1) [7, 9]. Bahar et al [7] by employing the Wilson-θ method, which is an unconditionally stable method, changed Eq. (1) in a manner that all responses of the controlled structure in the next time have been defined as a function of the responses in the instant time. Here, the procedure is briefly outlined consistent with the subsequent derivations of the control algorithm:

$$K_w \ddot{x}_{k+\theta} = p_{k+\theta} - K x_k - s_1 \dot{x}_k - s_2 \ddot{x}_k \tag{20}$$

and,

$$\begin{aligned} \ddot{x}_{k+1} &= \ddot{x}_k + e_3 (\ddot{x}_{k+\theta} - \ddot{x}_k) \\ \dot{x}_{k+1} &= \dot{x}_k + e_4 (\ddot{x}_{k+1} + \ddot{x}_k) \\ x_{k+1} &= x_k + \Delta t \dot{x}_k + e_2 (\ddot{x}_{k+1} + 2 \ddot{x}_k) \end{aligned} \tag{21}$$

where all of the parameters used in Eqs. (18) and (19) are defined as

$$\begin{aligned} & h = \theta \Delta t,\ e_0 = h/2,\ e_1 = h^2/6,\ e_2 = \Delta t^2/6, \\ & e_3 = 1/\theta,\ e_4 = \Delta t/2,\ K_w = M + e_0 C + e_1 K, \\ & s_1 = C + hK,\ s_2 = e_0 C + 2 e_1 K \end{aligned} \tag{22}$$

and

$$\begin{aligned} p_{k+\theta} = & (1-\theta)(B u_k + M e \ddot{x}_{0(k)}) + \\ & \theta (B u_{k+1} + M e \ddot{x}_{0(k+1)}) \end{aligned} \tag{23}$$

In this procedure, θ is 1.4. The time-dependent performance index, *J*(*t*), were modified to enhance serviceability of the structural system for occupant's comfort by adding the acceleration term, as follows

$$\begin{aligned} J(t) = \frac{1}{2} (& x_{k+1}{}^T Q_1 x_{k+1} + \dot{x}_{k+1}{}^T Q_2 \dot{x}_{k+1} + \\ & \ddot{x}_{k+1}{}^T Q_3 \ddot{x}_{k+1} + u_{k+1}{}^T R u_{k+1}) \end{aligned} \tag{24}$$

in which $n \times n$ positive semi-definite weighting matrices, $Q_i s$, are respectively related to displacement, velocity, and acceleration of the entire structure. By minimizing the time-dependent performance index, *J*(*t*), subject to discrete form of the equation of motion at each time instant, the control force vector, *u*(*t*), is generated by complete feedback of displacement, velocity, and acceleration vectors alone as follows:

$$\begin{aligned} u_{k+1} = & -R^{-1} B^T K_w{}^{-T} \times \\ & (Q_3 \ddot{x}_{k+1} + e_4 Q_2 \dot{x}_{k+1} + e_2 Q_1 x_{k+1}) \end{aligned} \tag{25}$$

where superscript (-*T*) means transpose of inverse matrix. Dependency of the instantaneous optimal control force vector using the Wilson-θ method on the time interval is apparent by presence of parameters *e*2, *e*4, and K_w in Eq. (22).

2.5. Discrete Instantaneous Optimal Control (DIOC)

Bahar et al [8] employs digital state space representation of Eq. (2), have introduced Discrete Instantaneous Optimal Control (*DIOC*) method as follows:

$$Z_{k+1} = A_d Z_k + B_d u_k + W_k \ddot{x}_g \quad , \quad Z(t_0) = Z_0 \tag{26}$$

where A_d, B_d, and W_d are transition matrices corresponding to A, B and H respectively and are defined as follows:

$$A_d = exp(A\Delta t) \quad , \quad F = \int_0^{\Delta t} exp(A\eta)\, d\eta$$
$$B_d = FB \quad , \quad W_d = FH \tag{27}$$

Bahar et al [8-9] have presented a new definition for quadratic time dependent performance index in discrete form as follows:

$$J(t) = \frac{1}{2}(Z_{k+1}{}^T Q Z_{k+1} + u_k{}^T B u_k) \tag{28}$$

This means decreasing energy supply for active control system at the instant time to achieve the best decreased responses of the structure in the next time. By minimizing $J(t)$ subjected to the constraint of Eq. (26), the control force vector is obtained as follows:

$$u_k = -[R + B_d{}^T Q B_d]^{-1} B_d{}^T Q (A_d Z_k + W_d \ddot{x}_{gk}) \tag{29}$$

In order to ensure the stability, Bahar et al [8-9] proposed a procedure based on the Lyapunov stability method in discrete form. In their proposed method, the stable discrete weighting matrix Q is computed from the following equation:

$$A_d Q A_d - A_d{}^T Q B_d (R + B_d{}^T Q B_d)^{-1} B_d{}^T Q A_d - Q + I_0 = 0 \tag{30}$$

In which I_0 is an arbitrary positive semi-definite matrix. It can be seen that this equation is also the matrix Riccati equation, which uses the discrete form of the controlled structure.

3. Simplified LQR Method (SLQR)

Assume that the previously defined Lagrange multiplier in Eq. (6) is not a time-dependent variable. This may not be true because it is directly related to the state vector, and we know that the state vector is not a constant function. In reality, this assumption is an irrational statement, but it is a little better than the assumption of zero external force during control time in Eq. (9). Now, if we accept this assumption, the first derivative of the Lagrange multiplier with respect to time is set to zero, and we have:

$$\lambda(t) = -2A^{-T} Q Z(t) \tag{31}$$

Substitute Eq. (31) in Eq. (7), we obtain:

$$u(t) = R^{-1} B^T A^{-T} Q Z(t) \tag{32}$$

This is a simple solution for Eq. (9) in continuous time based on the assumption of Eq. (31). In this method, named Simplified Linear Quadratic Regulator (*SLQR*), control force is directly related to the weighting matrix. Extensive analysis shows that this control force, because of its unusual assumption, neither guarantees the stability of the structure nor presents high efficiency of the control system. Hence, selecting proper and stable weighting matrices is a vital and necessary solution for the problem.

As mentioned before, Yang et al [9] by using the Lyapunov stability theory satisfied the necessary and sufficient conditions for the Instantaneous Optimal Control method (*IOC*). Similar condition is also satisfied by the discrete Instantaneous Optimal Control method, *DIOC*, results in the Discrete Stable Weighting (*DSW*) matrix [8]. Now, in order to ensure about the stability of *SLQR* and providing a good performance, finding a proper stable Q weighting matrix is inevitable.

Based on the Lyapunov direct method, a system is stable if a scalar Lyapunov function $V(Z) > 0$ for $Z \neq 0$, also $V(Z) = 0$ for $Z = 0$, and $V(Z) \to \infty$ as $Z \to \infty$ exists such that its first derivative with respect to time is negative definite for all Z, i.e. $\dot{V} < 0$. Based on this assumption, consider a positive definite matrix P, such as:

$$V^T(Z) P V(Z) > 0 \tag{33}$$

which is a possible Lyapunov function. Consider P is the coefficient of $Z(t)$ in Eq. (31) and replace it in Eq. (33), the Lyapunov function becomes as follows:

$$-2V^T(Z) A^{-T} Q V(Z) > 0 \tag{34}$$

First derivative of the Lyapunov function considering Eq. (2) and (31) is given as follows:

$$\dot{V}(Z) = Z^T(t)[-A^T A^{-T} Q - Q^T A^{-1} B R^{-1} B^T A^{-T} Q - A^{-T} Q A^T - A^{-T} Q B R^{-1} B^T A^{-T} Q] Z(t) \tag{35}$$

If the weighting matrix Q is selected such that the bracket in Eq. (35) to be negative definite, it is a stable weighting matrix. As a sufficient condition, we can assume that the sum of terms of the bracket in Eq. (35) is equal to a negative definite matrix $-I_0$ where I_0 is an arbitrary positive definite matrix. Using this definition, we get:

$$A^T A^{-T} Q + Q^T A^{-1} B R^{-1} B^T A^{-T} Q + A^{-T} Q A^T + A^{-T} Q B R^{-1} B^T A^{-T} Q - I_0 = 0 \quad (36)$$

If we replace $A^{-T}Q$ in the above equation by $\overline{Q}$ the following equation is obtained:

$$A^T \overline{Q} + \overline{Q}^T B R^{-1} B^T \overline{Q} + \overline{Q} A^T + \overline{Q} B R^{-1} B^T \overline{Q} - I_0 = 0 \quad (37)$$

and by selecting $\overline{Q}$ as a symmetric matrix, a Riccati type matrix equation is attained as follows:

$$A^T \overline{Q} + 2\overline{Q} B R^{-1} B^T \overline{Q} + \overline{Q} A^T - I_0 = 0 \quad (38)$$

Now if positive definite matrix I_0 is properly selected and Eq. (38) is solved for matrix, $\overline{Q}$ stable matrix Q may be obtained from $A^{-T}Q$.

4. Numerical Example

In order to investigate performance of different control methods and specially the proposed one, an eight-story shear-type building structure is considered. Properties of the building are defined as follows: the floor mass of each story is 345.6 tons, the elastic stiffness of each story is 3.404×10^5 *kN/m*, and the internal damping coefficient of each story is 2937 *tons-sec/m*. The input excitation is the *N-S* component of the 1940 El-Centro (Imperial Valley) earthquake record with 0.33 *g* peak ground acceleration and about 54 seconds time length. Characteristics of the implemented active mass damper/driver (*AMD*) system on the top floor are as follows: The mass is 29.63 *tons*, the tuned frequency is 98% of the first vibration frequency of the building, and the damping is 25 *tons-sec/m*.

In addition, as a passive system, a tuned mass damper (*TMD*) with similar dynamic specifications without needing external energy or inserting control force to the building implemented on the top floor. Effectiveness of the various optimal control algorithms is investigated in two different cases by comparing the maximum building responses in different conditions: without a control system, equipped with passive control system, and equipped with active control systems using different control laws. Two different cases are separated by using (I) general selected weighting matrices, (II) stable weighting matrices. The control algorithms are used in comparison includes six control methods as: (1) the classic closed-loop optimal control, *CCLQR*, (2) the instantaneous optimal control, *IOC*, (3) the generalized optimal active control, *GOAC*, (4) the instantaneous optimal Wilson-θ control method, (5) the discrete instantaneous optimal control, *DIOC*, and (6) the new proposed simplified linear quadratic regulator method, *SLQR*. To achieve an acceptable and reliable judgment, we imposed that the average required control force (*ACF*) for all control systems is the same. It seems this is the only way that comparing between performances of different control systems is admissible. *ACF* is determined by the integration of the absolute value of the entire instant control forces over the time divided by the control time duration. It is assumed to be equal to 72.68 *kN*. It means the power-supply system for all control systems is identical, but the instant maximum control forces of the control systems are not necessarily the same.

4.1. Case I: Selecting General Weighting Matrices

In case I, all the control algorithms work with general selected weighting matrices. In other words, different kinds of symmetric positive semi-definite matrices, as Q matrix, have been investigated and the matrix presenting the best performance in the specific value of *ACF* is selected. This is a cumbersome and time consuming procedure because it needs continual analysis by different matrices or various parameters for a proper selected matrix. The selected weighting matrices are as follows:

IOC Method. Yang's proposed weighting matrix is used for *IOC* method [6], except that we have changed it to a symmetric form and multiplied it to 25.271. The elements of this matrix are as follows:

$$Q = \begin{bmatrix} 0 & E & S \\ E^T & K_{11} & K_{12} \\ S^T & K_{22} & K_{22} \end{bmatrix} \quad (39)$$

in which the submatrices are as follows: E^T=[-33.5, -67, -100.5, -134, -167.5, -201, -234.5, 268, 375, 67.5, 135, 202.5, 270, 337.5, 405, 425.5], S^T=[-33.5, -67, -100.5, -134, -167.5, -201, -234.5, -268, 32.2, 5.8, 11.6, 17.4, 23.2, 29, 34.8, 40.6], and K_{11}=540, K_{12}=K_{21}= 32.2, K_{22}=5.7.

GOAC and DIOC Methods. General weighting concern matrices with *GOAC* and *DIOC* methods are defined as follows:

$$Q = \begin{bmatrix} \alpha K_\gamma & 0_{9\times 9} \\ 0_{9\times 9} & \beta M \end{bmatrix}$$

$$K_\gamma = \begin{bmatrix} K_{11}(8\times 8) & 0_{9\times 9} \\ 0_{9\times 9} & \beta M \end{bmatrix} \quad (40)$$

where K_γ is the entire stiffness matrix of the building, just γ has been substituted by a small number instead of the $K99$ entry of this matrix. In order to obtain the required *ACF* and proper performance, the coefficients α, β, and γ are respectively defined as 1×10^4, 92.025 and 21.

Wilson-θ Method. This method needs to define three different characteristic-related weighting matrices as follows:

$$Q_1 = 10\begin{bmatrix} I_{8\times 8} & 0 \\ 0 & -0.001 \end{bmatrix}$$

$$Q_2 = 10^8\begin{bmatrix} I_{8\times 8} & 0 \\ 0 & 0.001 \end{bmatrix} \quad (41)$$

$$Q_3 = \beta I_{9\times 9}$$

where Q_1, Q_2 and Q_3 are acceleration, displacement and velocity related weighting matrices. Coefficient β is equal to 8920 and R is selected such that *ACF* reaches to the desired value.

SLQR Method. The following weighting matrix selected for *SLQR* method presents the best possible performance without indication of instability in the controlled building:

$$Q = \begin{bmatrix} I_{9\times 9} & 0_{9\times 9} \\ 0_{9\times 9} & 0_{9\times 9} \end{bmatrix} \quad (42)$$

in which I is the unit matrix related to the displacement response of the building. Defining a velocity-related coefficient matrix in Eq. (43), results in instability.

Matrix R. R matrix is defined such that the average control forces of all the mentioned control systems are adjusted to the specified value. The values of R matrix for different optimal control methods are tabulated in Table (1). Because, there is only one *AMD* installed at the 8^{th} story, R matrix is a number.

Table 1. The values of R for different optimal control methods using a general weighting matrix.

IOC	GOAC	Wilson-q	DIOC	SLQR
0.00101	0.0014	0.001	0.00095	0.652

4.2. Case II: Selecting Stable Weighting Matrices

In order to find stable weighting matrices for each optimal control method, their concerned Riccati matrix equation should be solved. Coefficients of the Riccati matrix equations for each method are composed of special continuous or discrete matrices of the system. By solving the equation, an exclusive stable weighting matrix for each method will be obtained.

CCLQR and GOAC Methods. The weighting matrix Q is selected as follows:

$$Q = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \quad (43)$$

where K and M are the matrices with the dimensionless numerical values of the stiffness and mass matrices of the controlled building, neglecting the stiffness and mass values of the active mass damper/driver. Extensive analysis shows that such arrangement of weighting matrix results in higher performance of the control system than selecting unit matrix arrangement, especially when the average control forces are limited [8].

IOC, and DIOC Methods. For other optimal control algorithms in order to determine a proper stable matrix, there is a common positive definite matrix, named I_0, which must be defined a priori. We have found that this arrangement is a proper matrix for achieving the stable matrix:

$$I_0 = \begin{bmatrix} \alpha K_\gamma & 0_{9\times 9} \\ 0_{9\times 9} & \beta M \end{bmatrix} \text{ where } K_\gamma = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & \gamma \end{bmatrix} \quad (44)$$

where M is mass matrix of the entire controlled structure, and $K\gamma$ is constructed using submatrices of the entire stiffness matrix of the building, γ is an arbitrary scalar value related to the driver mass stiffness. The value γ is selected as small as the Kg is still a positive definite matrix. Coefficients α, β, γ and also the value of R matrix are found such that efficiency of the control system is high and the average required control force is equal to the specified value. Based on this assumption, for *IOC* method, the coefficients of α, β, and γ are respectively assigned to 876.2, 100, and 21. Similarly, these parameters for *DIOC* method are equal to 4×10^3, 32.275, and 21, respectively.

SLQR Method. In order to achieve proper performance of *SLQR* method, the positive definite matrix I_0 is chosen similar to the definition of matrix Q in Eq. (43). Hence, appropriate $\overline{Q}$ and then Q is determined as ATQ.

Matrix R. The values of R matrix for different optimal control methods using stable weighting matrices are tabulated in Table (2).

Table 2. The values of R for different optimal control methods using a stable weighting matrix.

CCLQR / GOAC	IOC	DIOC	SLQR	CCLQR /GOAC
0.0003765	0.0835	0.88	0.000393	0.0003765

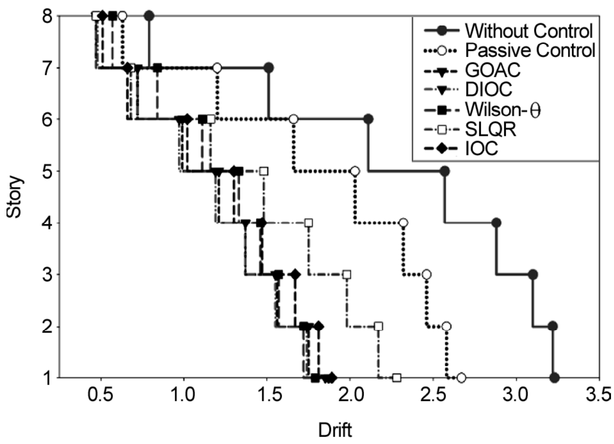

Figure 2. Maximum drift ratios of the floors, comparing between different optimal control algorithms using general selected weighting matrices.

5. Results and Discussion

Linear time history analysis for various algorithms and related weighting matrices are carried out. In each case, maximum responses of the building and control system are determined. Take notice that all the average control forces are the same. It means that all the control systems need a similar amount of energy supply, although the maximum control forces may be different.

5.1. Response of the Building

Case I. Maximum responses, displacement, velocity and acceleration of the floors of the building controlled by different optimal control algorithms using general selected weighting matrices are compared together in Figure (1). Also, results of the maximum story drift ratios are shown in Figure (2). It can be easily recognized that maximum floor responses of the building equipped with a passive mechanism are slightly better than the building without control,the reason of which, in this case, may be considered as the tuning of the control system for an effective active rather than passive action.

Extensive efforts show that finding proper general weighting matrix for *SLQR*, which guarantee stability of the controlled building, is very difficult. Without having a plan to form such weighting matrix, the controlled building is highly vulnerable to instability, and efficiency of the control system is very low. As it may be seen in Figure (1), results of *SLQR* are only a little better than that of the passive control system. It means that, this method without

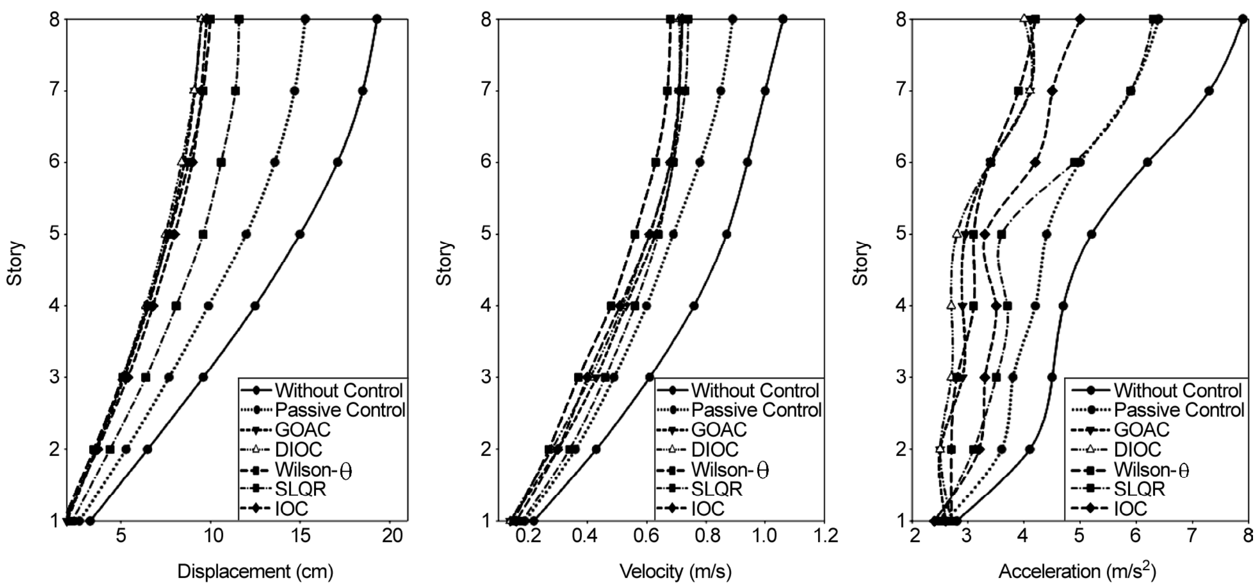

Figure 1. The maximum responses of the floors, comparing between different optimal control algorithms using general selected weighting matrices.

using stable weighting matrix if it produces a stable condition needing a large amount of control forces to be efficient. The result for *SLQR* is certainly acceptable because this method has not a logical procedure to determine control force.

On the other hand, the other methods, i.e. *IOC*, *GOAC*, Wilson-θ, and *DIOC* using general selected weighting matrices perform such that their efficiency in controlling all the responses of the building are acceptable. In Figure (2), comparing results of the maximum drift ratios obtained from different optimal control algorithms confirm again the above-mentioned results. It is mentioned that in Case I, stability criterion should be carefully observed by the designer. In other words, there are infinite general weighting matrix, which may produce instability of the building. Hence, a designer will never really be sure of selecting a proper weighting matrix for different conditions or excitations.

By comparing maximum responses of the floors in Case I, *DIOC* algorithm using general weighting matrix is recognized as the best control method to decrease building responses. Results of this method as a limit for using general weighting matrix are compared with the methods in Case II.

Case II. In Figure (3) the maximum responses of the floors due to performing different optimal control algorithms using determined stable weighting matrices are compared together and with the best of the previous case. Maximum drift ratios are shown in Figure (4). Results show that *DIOC* method using

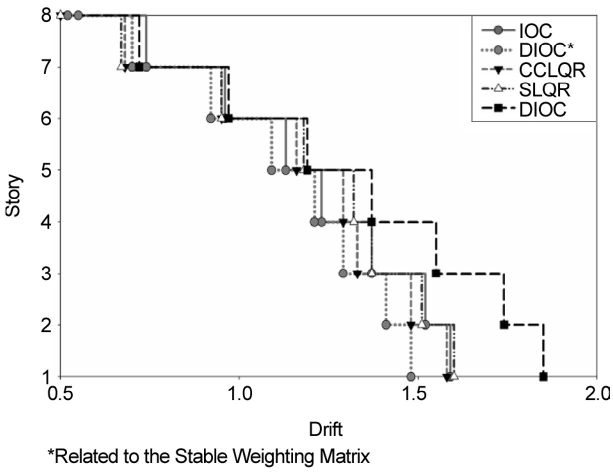

Figure 4. Maximum drift ratios of the floors, comparing between different optimal control algorithms using stable weighting matrices.

general weighting matrix is very good in controlling acceleration responses, but all the optimal control methods using stable weighting matrix are more successful in controlling displacement responses. As mentioned before, *GOAC* by using a stable matrix like *P* matrix is completely the same as *CCLQR*. Therefore, it may be said that all the family of optimal control algorithm either classic, general, or instantaneous branch by using a determined stable weighting matrix perform very close together such that their differences are neglegible. Results shown in Figure (4) confirm the above-mentioned results, too.

Although differences are very small, it seems the

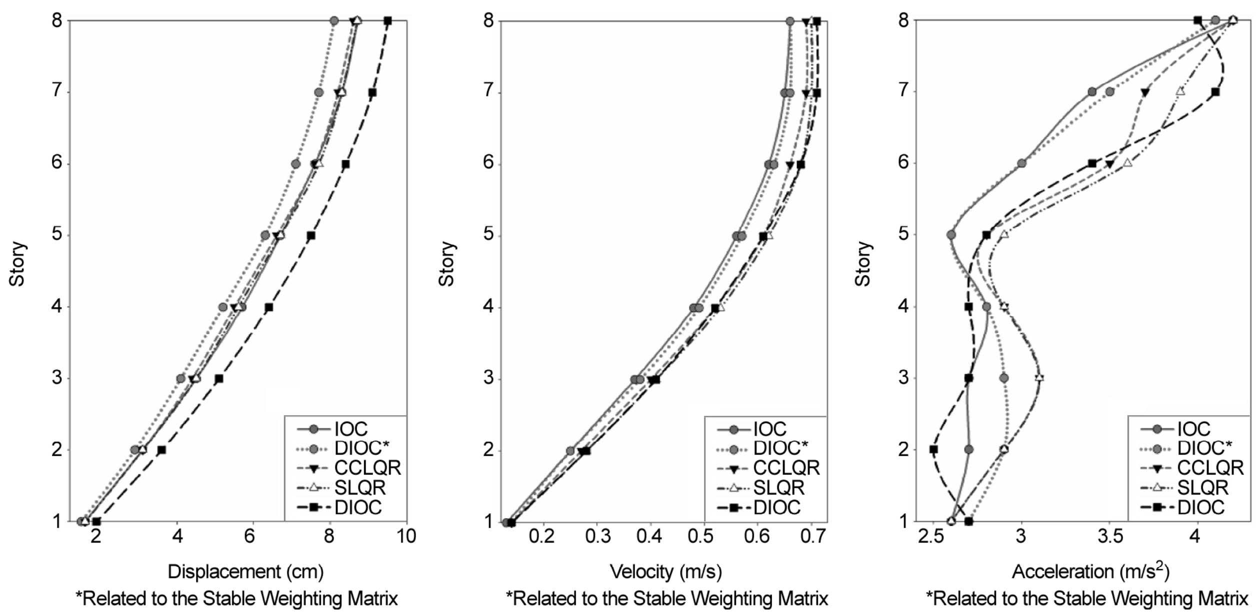

Figure 3. The maximum responses of the floors, comparing between different optimal control algorithms using general selected weighting matrices.

efficiency of *DIOC* method is the highest. The efficiency of *IOC* using stable weighting matrix is very close to that of *DIOC* method specially it performs slightly better in decreasing acceleration responses. Performance of the new proposed method, *SLQR*, is completely similar to *CCLQR* method. It is strange because *SLQR* method has not proposed based on logic criteria, but it is designed by an irregular assumption. So, well behavior of the controlled building using *SLQR* algorithm is rooted in satisfying stability criteria to obtain their specific stable weighting matrices. In this view, all mentioned optimal methods of Case II have a common foundation such that they should solve the matrix Riccati eqation either during their procedure like *CCLQR* to determine *P* matrix, or during determining stable weighting matrices like *IOC*, *GOAC* (when using *P* matrix), *DIOC*, and *SLQR*. In other words, all the discussed optimal procedures presented here in spite of their various basic assumptions to determine control forces by solving their concern matrix Riccati equations present robust control systems with high efficiency such that they guarantee stable responses of the controlled building.

Base Shear. Maximum base shear reduction of the building due to earthquake excitation for various passive and active control systems are determined. Also, their reductions with respect to the base shear of the building without control system using general weighting matrices are determined and presented in the second column of Table (3). Results show that the best performance of the base shear reduction is related to the Wilson-θ method, about 45%, and the worst performace belongs to *SLQR*, about 30%. Base shear reduction of control system using stable weighting matrix are presented in the second column of Table (4). It is clear that all the methods have very good performances among *SLQR* method presents 51% reduction. *DIOC* method with 55% reduction in base shear of the controlled building is the best.

5.2. Response of the Active Driver Mass

The maximum control force and driver mass responses for different control systems using general or stable weighting matrices are respectively presented in the columns 3 to 6 of Tables (3) and (4). Since, the passive system is excited by the responses of the building, the driver mass responses are much smaller in comparison with other systems. Although all the average control forces (*ACF*) of control mechanisms are the same, they have different maximum control forces. This need is only likely for a fraction of a second, but the control system should have potential for generating this control force. Therefore, the control systems need more maximum control force, may be naturally more expensive than the others. In this view, it seems that providing energy supply for *IOC* and *SLQR* algorithm are cheaper, but responses of the control system are too high to be efficient. Also, the Wilson-θ method needs large amount of energy to generate such large instant control force. Hence, *GOAC* and *DIOC*

Table 3. The maximum values of base shear reduction and control system responses by use of general selected weighting matrices.

Used Method	Max Base Shear Reduction (%)	Max Control Force (kN)	Max Driver Mass Responses		
			Displacement (m)	Velocity (m/s)	Acceleration (m/s^2)
Passive Control	16.60	---	0.61	3.61	19.90
IOC	42.20	655.13	1.73	10.76	64.60
GOAC	43.20	735.00	1.17	6.64	48.10
DIOC	43.00	738.61	1.17	6.65	47.80
Wilson-θ	44.60	810.08	1.22	6.11	42.20
SLQR	29.90	613.79	1.95	11.53	71.00

Table 4. The maximum values of base shear reduction and control system responses by use of stable weighting matrices.

Used Method	Max Base Shear Reduction (%)	Max Control Force (kN)	Max Driver Mass Responses		
			Displacement (m)	Velocity (m/s)	Acceleration (m/s^2)
CCLQR/GOAC	51.50	702.96	1.46	8.41	49.60
IOC	51.30	661.68	1.32	7.38	50.50
DIOC	54.70	637.13	1.43	8.29	48.90
SLQR	50.50	713.05	1.48	8.51	50.90

methods with a reasonable maximum control force and smaller driver mass responses are acceptable control algorithms for Case I.

Generally speaking, when control systems using stable weighting matrices, Case II, the need for maximum control force are reduced but control system responses are increased. In this regard, performances of the control systems are very close together. The best performance belongs to *IOC* method, then *DIOC* method and after that other methods have the same performance. Performance of *SLQR* and *CCLQR* methods are very similar. Similarities of these two methods are emphasized on the mentioned fact: solving the matrix Riccati equation to determine stable weighting matrix as a coefficient of the gain matrix is more important than the criteria used to define control force.

5.3. Frequency Response Function

Frequency response function (*FRF*) of a controlled system shows location of its main frequencies and its sensitivity (amplification or attenuation) to the external disturbances at a glance. In Figure (5), *FRF* plots of the building controlled by algorithms of Case I are shown. Two specific behaviors are recognizable. The first one emerged in Passive, *SLQR*, and *IOC* methods, divide energy of the first mode into two distant separate observable modes with lesser power. The second one emerged in *GOAC*, Wilson-θ, and *DIOC* methods, alleviate the power of both first and the second modes of the building. This difference returns to the element arrangements of the selected weighting matrices. Extensive analysis shows that the second group presents also more decrease in power of the second mode. Referring to Figure (1) it is seen that more effectiveness of a control system on the higher modes results in smaller acceleration responses of the building.

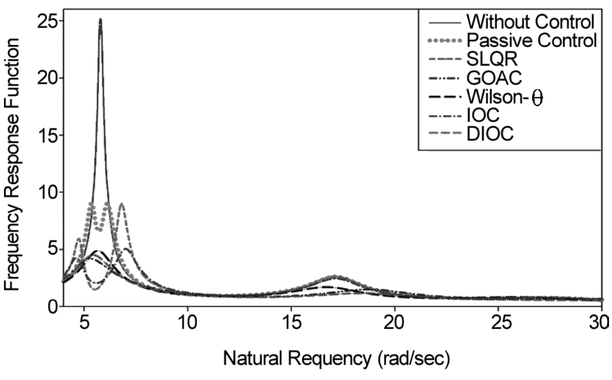

Figure 5. Frequency responses function of the controlled building, comparing between different optimal control algorithms using general selected weighting matrices.

In Case II, behaviors of the different control systems through using the stable weighting matrices are very close together, Figure (6). It is seen that amplitude of the first mode concern to all methods are smaller than that of *DIOC* with general weighting matrix. Therefore, we expect that responses of the floors are at least slightly smaller for Case II, refer to Figure (3). On the other hand, effectiveness of *DIOC* with general weighting matrix on the 2^{nd} and 3^{rd} mode amplitudes is recognizable in acceleration responses of the building, refer to Figure (3). However, this is not the only effect, because both *DIOC** and *IOC* present good behavior on the acceleration responses control especially for higher floors. Actually, moving frequency location besides decreasing its amplitude may alter the behavior of the building. In addition, this matrix is not easily accessible unless checking all the alternate definitions of weighting matrices for a special building.

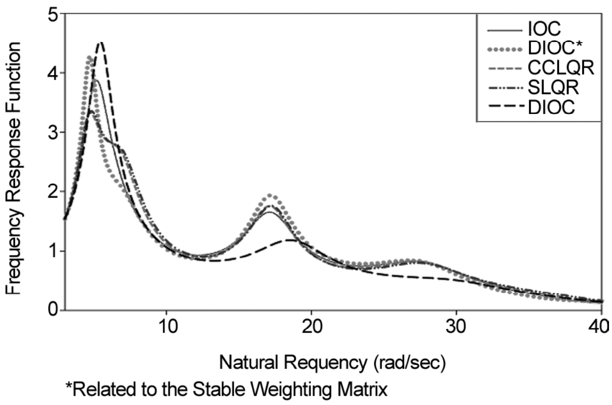

Figure 6. Frequency responses function of the controlled building, comparing between different optimal control algorithms using stable weighting matrices.

5.4. Stability Criteria

In order to consider stability margin of the controlled building in Figure (7) complex frequencies of the system matrix concern to the considered building without control, with passive control, and with different active controls are compared together. They all show good performances. Their marginal stability, i.e. the distance of the frequencies location to the unit circle from the control systems are very close, too. The only difference is for *DIOC* using general weighting matrix. It has completely changed

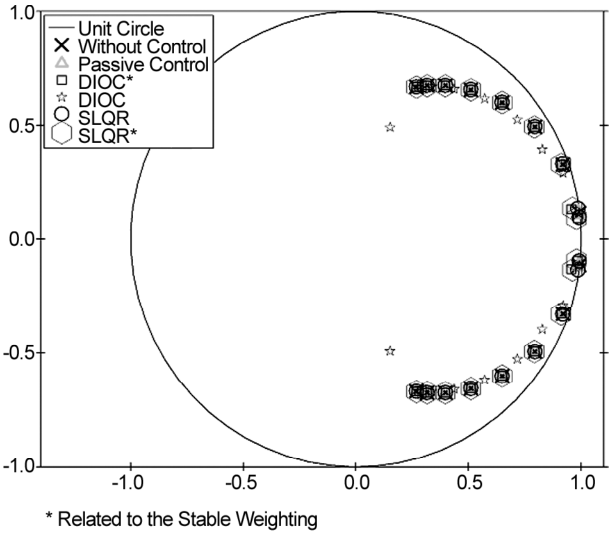

Figure 7. Stability diagram of the controlled building, comparing between different optimal control algorithms using general or stable weighting matrices.

the location of all the frequencies of the primary building inside the unit circle. Hence, one goal for a designer may be selecting general weighting matrices such that they move the main frequencies of the controlled building inside the unit circle as much as possible.

As a special view to *SLQR* method, it seems that *SLQR* moves the complex frequencies of the controlled building more inside the unit circle. This means *SLQR* method using stable weighting matrix produce large enough marginal stability of the controlled building such that it may perform well against external loadings. Robustness of the control system is not related to their assumption but to their guarantee of satisfying extra stability criteria. Thus, the building equipped with an active mass driver/damper using *SLQR* algorithm is stable enough to act against strong ground accelerations.

6. Conclusions

There are many optimal control algorithms, which some of them like the classic optimal control, the generalized optimal active control and the instantaneous optimal Wilson-θ control method are inherently robust control methods. But, in some of them, like the instantaneous optimal control or discrete instantaneous optimal control method, stability criteria, or generating weighting matrices are not satisfied during determination of control forces. Therefore, performances of different optimal algorithms are evaluated in two separate cases. Case I concerns with the optimal control algorithms like *IOC*, *GOAC*, Wilson-θ, and *DIOC* methods using general selected weighting matrices. Case II, concerns with the optimal control algorithms like *LQR*, *IOC*, and *DIOC* methods using stable weighting matrices. These methods have a logic base and routine procedure to determine control forces. Extensive analysis shows that finding proper general weighting matrices for control methods with high performance such that it guarantees the stability of the controlled building are a cumbersome and time-consuming procedure. On the other hand, determining weighting matrix of the methods in Case II are much simpler and their performances are much better than that of Case I. Evaluations show that using stable weighting matrix by solving the matrix Riccati equation results in a very good performance of a control system. In order to evaluate this claim, an irrational and illogical procedure, named Simplified Linear Quadratic Regulator (*SLQR*), which neither guarantees stability of the controlled building nor presents high performance of the control system are proposed. Hence, by using the Lyapunov stability theory, a procedure to achieve stable weighting matrix, *Q*, is presented. Performance of *SLQR* in comparison with the other mentioned optimal control algorithms is examined. Extensive analysis shows that efficiency of the proposed method using a stable weighting matrix is very similar to the other mentioned methods. This is basically because the family of the optimal control methods solving the matrix Riccati equation either during their procedure, like *LQR* method, or during satisfying the stability criteria, like *IOC*, *DIOC* and *SLQR*, in spite of their assumption for determining control forces, guarantee robustness and stability of the controlled structure and present high efficiency of the control system.

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